1st Order Linear D.E.

General Comments:
Given the differential equation,

$$y' + p(t)y = g(t)$$

the solution takes the form:

$$y(t) = c_1 y_1$$

where y_1 is obtained from using the integrating factor method.

Derivation:
Define the integrating factor as:

$$\mu = e^{\int p(t)dt}$$

Observe that taking the derivative with respect to t yields:

$$\mu' = \frac{d}{dt}(\int p(t)dt)e^{\int p(t)dt} = p(t)e^{\int p(t)dt} = p(t)\mu$$

Distributing the integrating factor through the differential equation reveals an internal product rule on the left hand side of the equation.

$$\mu(y' + p(t)y) = \mu g(t)$$

$$\mu y' + (\mu p(t))y = \mu y' + \mu' y = \mu g(t)$$

$$(y\mu)' = \mu g(t)$$

Integrating the sides with respect to t and dividing by the integrating factor solves the differential equation

$$\int (y\mu)' dt = \int \mu g(t) dt$$

$$y\mu = \int \mu g(t) dt$$

$$y = \frac{1}{\mu} \int \mu g(t) dt$$

Solution Steps:

$$y' + p(t)y = g(t)$$

$$\mu = e^{\int p(t)dt}$$

$$y = \frac{1}{\mu} \int \mu g(t) dt$$

Exact Equations

General Comments:
Given the differential equation,

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

the method of exact equations can be used to solve for $y(x)$

Testing Exactness:
If the function is exact in its current state, then

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \text{ and } \frac{d\mu}{dx} = 0$$

otherwise an integrating factor $\mu \neq 0$ is required. This integrating factor can take two forms depending on assumption that it is purely a function of x or y:

$$\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu = -\frac{M_y - N_x}{M}\mu$$

which yields

$$\frac{\partial \mu M}{\partial y} = \frac{\partial \mu N}{\partial x}$$

Solution Method:

$$\mu(M(x,y) + N(x,y)\frac{dy}{dx}) = \mu M(x,y)dx + \mu N(x,y)dy = 0$$

$$\Psi(x,y) = \int (\mu M(x,y)dx + \mu N(x,y)dy) = \int 0dx = c$$

$$\Psi(x,y) = \int \mu M(x,y)dx + \int \mu N(x,y)dy = c$$

Removing Redundant Terms:
The resulting solution equation Ψ must be such that, omitting constants C,

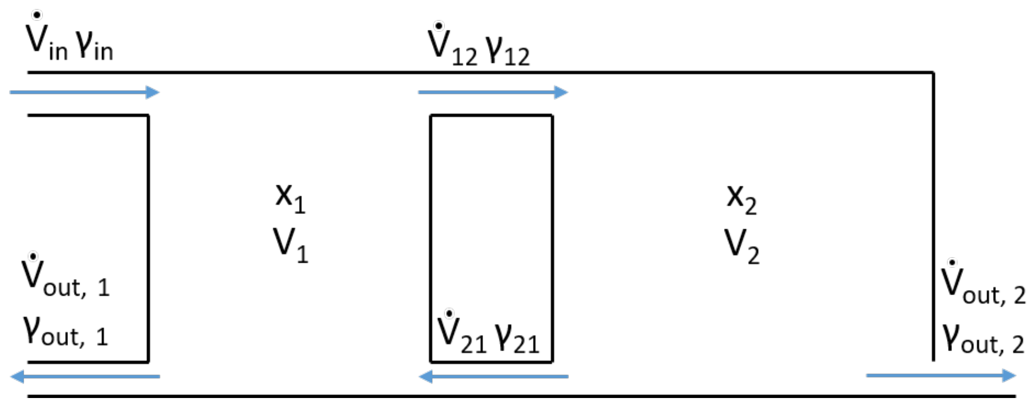
$$\Psi(x,y) = \int \mu M(x,y)dx = \int \mu N(x,y)dy$$

which allows for the reduction of redundant terms.

Tank Mixture Problems

General Comments:
Given a tank problem drawn below where $x_i(t)$ is the amount of substance mixed with water, V_i is the volume of the tank, the solution is as follows:

Flow Diagram.png



General Differential Equation:
For the tank system, a differential equation for each tank is written as the rate of mixture in minus the rate of mixture out. Note the concentrations (γ) and flow rate ($\dot{V}$) of each stream leaving and entering the tank.

$$\frac{d(x_i)}{dt} = \Sigma(\gamma\dot{V})_{in} - \Sigma(\gamma\dot{V})_{out}$$

Note that fluid leaving a "well stirred tank" has concentration $\gamma_{well-stirred} = \frac{x_i(t)}{V_i}$

Solution Method: Summing all the concentrations and flow rates in and out of tank 1 and two yields the following system of equations:

$$\frac{d(x_1)}{dt} = \gamma_{in}\dot{V}_{in} + \gamma_{21}\dot{V}_{21} - \gamma_{out,1}\dot{V}_{out,1} - \gamma_{12}\dot{V}_{12}$$

$$\frac{d(x_2)}{dt} = \gamma_{12}\dot{V}_{12} - \gamma_{21}\dot{V}_{21} - \gamma_{out,2}\dot{V}_{out,2}$$

Replacing the concentration terms with their well-stirred terms:

$$\frac{d(x_1)}{dt} = \gamma_{in}\dot{V}_{in} + \frac{x_2}{V_2}\dot{V}_{21} - \frac{x_1}{V_1}\dot{V}_{out,1} - \frac{x_1}{V_1}\dot{V}_{12} = \gamma_{in}\dot{V}_{in} + x_2\frac{\dot{V}_{21}}{V_2} - x_1(\frac{\dot{V}_{out,1} + \dot{V}_{12}}{V_1})$$

$$\frac{d(x_2)}{dt} = \frac{x_1}{V_1}\dot{V}_{12} - \frac{x_2}{V_2}\dot{V}_{21} - \frac{x_2}{V_2}\dot{V}_{out,2} = x_1\frac{\dot{V}_{12}}{V_1} - x_2(\frac{\dot{V}_{out,2} + \dot{V}_{21}}{V_2})$$

Note: If the tank system has only one tank, only the first equation is required with $\dot{V}_{21} = \dot{V}_{12} = 0$

Tank Mixture Problems (Cont.)

Elimination Method:
Similar to how linear systems of equations were solved in College Algebra, adding and subtracting the differential equations here will be used to reduce the equation to a function of only one variable.
First each eqation is set equal to 0:

$$\frac{d(x_1)}{dt} = \gamma_{in}\dot{V}_{in} + x_2\frac{\dot{V}_{21}}{V_2} - x_1(\frac{\dot{V}_{out,1} + \dot{V}_{12}}{V_1}) \rightarrow \frac{d(x_1)}{dt} - \gamma_{in}\dot{V}_{in} - x_2\frac{\dot{V}_{21}}{V_2} + x_1(\frac{\dot{V}_{out,1} + \dot{V}_{12}}{V_1}) = 0$$

$$\frac{d(x_2)}{dt} = x_1\frac{\dot{V}_{12}}{V_1} - x_2(\frac{\dot{V}_{out,2} + \dot{V}_{21}}{V_2}) \rightarrow \frac{d(x_2)}{dt} - x_1\frac{\dot{V}_{12}}{V_1} + x_2(\frac{\dot{V}_{out,2} + \dot{V}_{21}}{V_2}) = 0$$

Let $D = \frac{d}{dt}$ such that $D[x_i] = \frac{x_i}{dt}$ and factor out the x_1, x_2 terms as follows:

$$(D + (\frac{\dot{V}_{out,1} + \dot{V}_{12}}{V_1}))[x_1] - \gamma_{in}\dot{V}_{in} - x_2\frac{\dot{V}_{21}}{V_2} = 0$$

$$(D + (\frac{\dot{V}_{out,2} + \dot{V}_{21}}{V_2}))[x_2] - x_1\frac{\dot{V}_{12}}{V_1} = 0$$

Multiplying equation one by $\frac{\dot{V}_{12}}{V_1}$ and equation two by $(D + (\frac{\dot{V}_{out,1}+\dot{V}_{12}}{V_1}))$ allows us to cancel the x_1 terms through addition of the two equation and results in the differential equation:

$$(D + (\frac{\dot{V}_{out,1} + \dot{V}_{12}}{V_1}))(D + (\frac{\dot{V}_{out,2} + \dot{V}_{21}}{V_2}))[x_2] - \frac{\dot{V}_{12}}{V_1}(\gamma_{in}\dot{V}_{in} + x_2\frac{\dot{V}_{21}}{V_2}) = 0$$

$$(D^2 + (\frac{\dot{V}_{out,1} + \dot{V}_{12}}{V_1} + \frac{\dot{V}_{out,2} + \dot{V}_{21}}{V_2})D + \frac{(\dot{V}_{out,1} + \dot{V}_{12})(\dot{V}_{out,2} + \dot{V}_{21})}{V_1V_2})[x_2] - \frac{\dot{V}_{12}}{V_1}(\gamma_{in}\dot{V}_{in} + x_2\frac{\dot{V}_{21}}{V_2}) = 0$$

$$\frac{d^2(x_2)}{dt^2} + (\frac{\dot{V}_{out,1} + \dot{V}_{12}}{V_1} + \frac{\dot{V}_{out,2} + \dot{V}_{21}}{V_2})\frac{d(x_2)}{dt} + (\frac{(\dot{V}_{out,1} + \dot{V}_{12})(\dot{V}_{out,2} + \dot{V}_{21}) - \dot{V}_{12}\dot{V}_{21}}{V_1V_2})x_2 = \frac{\dot{V}_{12}}{V_1}\gamma_{in}\dot{V}_{in}$$

x_2 can be solved by use of the following methods of solving second order non-homogeneous differential equations.

Further, x_1 can be solved for by another elimination method or by substituting x_2, x_2' back into the above differential equation

Homogeneous Equations

General Comments:
Given the differential equation,

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = 0$$

the solution takes the form:

$$y(t) = c_1y_1 + c_2y_2$$

where y_1 and y_2 are obtained by solving the characteristic equation of the differential equation.

Characteristic Equation:
If $p(t)$ and $q(t)$ are constants, assume a solution of $y = e^{\lambda t}$ with first and second derivatives $\lambda e^{\lambda t}$ and $\lambda^2 e^{\lambda t}$ respectively.

Plugging this into the differential equation yields:

$$e^{\lambda t}(\lambda^2 + p(t)\lambda + q(t)) = 0$$

which is solved through the quadratic formula to yield homogeneous solutions y_1 and y_2.

$$\lambda_{1,2} = \frac{-p(t) \pm \sqrt{p(t)^2 - 4q(t)}}{2} \to y_1 = e^{\lambda_1 t},\, y_2 = e^{\lambda_2 t}$$

Repeated Roots:
If $\lambda_1 = \lambda_2$ then the repeated homogeneous solution is multiplied by t such that

$$y_1 = e^{\lambda t},\, y_2 = te^{\lambda t}$$

Complex Roots:
If $p(t)^2 - 4q(t) < 0$ such that the roots of the characteristic equation are complex, they take the form:

$$\lambda_{1,2} = \frac{-p(t)}{2} \pm \frac{\sqrt{4q(t) - p(t)^2}}{2}i = \eta \pm \omega i \to y_1 = e^{(\eta+\omega i)t},\, y_2 = e^{(\eta-\omega i)t}$$

by the Euler identity $e^{i\alpha} = cos(\alpha) + isin(\alpha)$ and rearranging of the variables, the solution becomes

$$y_1 = e^{\eta t}cos(\omega t),\, y_2 = e^{\eta t}sin(\omega t)$$

Nonhomogeneous Equations - Method of Undetermined Coefficients

General Comments:
Given the differential equation,

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = g(t)$$

the solution takes the form:

$$y(t) = c_1y_1 + c_2y_2 + y_p$$

where y_1 and y_2 are obtained by solving the differential equation as a homogeneous equation and method of undetermined coefficients is utilized to obtain y_p.

For the method of undetermined coefficients, y_p is defined based off of $g(t)$ and the solutions to the homogeneous case.

Choosing y_p's Form from $g(t)$:

-If $g(t)$ has a term t^n, then y_p should be assumed to contain a polynomial of degree n such that $y_p = (A_nt^n + A_{n-1}t^{n-1} + \ldots + A_0t^0) + h(t)$ where $h(t)$ is any other functions from this section.

-If $g(t)$ has a term $sin(\omega t)$ or $cos(\omega t)$, then y_p should be assumed as $y_p = (A_0cos(\omega t) + A_1sin(\omega t) + h(t)$ where $h(t)$ is any other functions from this section.

-If $g(t)$ has a term $e^{\alpha t}$, then y_p should be assumed as $y_p = A_0e^{\alpha t} + h(t)$ where $h(t)$ is any other functions from this section.

Choosing y_p's Form from Homogeneous Solutions:

-If the proposed solution created from the above $g(t)$ rules has a term that matches the form of any solutions to the homogeneous equation, the anticipated solution should be multiplied by t.

Solution of Coefficients:
Take all required derivatives of the proposed solution and plug in to the differential equation:

$$y_p'' + p(t)y_p' + q(t)y_p = g(t)$$

and solve for all constants in y_p.

Nonhomogeneous Equations - Variation of Parameters

General Comments:
Given the differential equation,

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = g(t)$$

the solution takes the form:

$$y(t) = c_1y_1 + c_2y_2 + y_p$$

where y_1 and y_2 are obtained by solving the differential equation as a homogeneous equation and variation of parameters is utilized to obtain y_p.

Particular Solution:
For variation of parameters, y_p is defined:

$$y_p = \nu_1 y_1 + \nu_2 y_2,\ \nu_1 = \int \frac{-g(t)y_2}{W}dt, \text{ and } \nu_2 = \int \frac{g(t)y_1}{W}dt$$

Note that ν_1 includes a negative sign but ν_2 does not.

with W in this case being the Wronskian.

The Wronskian can be calculated by the following formula:

$$W = det(\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = y_1y_2' - y_2y_1'$$

Note that the Wronskian cannot be zero as used in the integration above.

Reduction of Order:
If $y_1(t)$ is not identically 0, then the homogeneous solutions to the differential equation are related by

$$y_2 = y_1 \int \frac{e^{-\int p(t)dt}}{y_1^2}dt$$

Laplace Transforms

General Comments:
Given the differential equation and initial conditions,

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = g(t),\ y(0) = k_1,\ y'(0) = k_2$$

the solution takes the form:

$$y(t) = c_1y_1 + c_2y_2 + y_p$$

where y_1, y_2, and y_p are obtained by transforming the differential into the Laplace domain and returning the solution to the time domain.

Properties of the Laplace Transform:

-The Laplace Transform is a linear operator which means it can be "distributed" over terms being added or subtracted. I.e.

$$\mathcal{L}[A(t) + B(t)] = \mathcal{L}[A(t)] + \mathcal{L}[B(t)]$$

-Scalar multiples of functions undergoing Laplace Tranforms are as follows:

$$\mathcal{L}[cA(t)] = c\mathcal{L}[A(t)]$$

Laplace Transform of $y^{(n)}(t)$:
The Laplace Transform of any derivative of the solution function yields,

$$\mathcal{L}[y^{(n)}(t)] = s^nY(s) - s^{n-1}y(0) - s^{n-2}y'(0) - \ldots - y^{n-1}(0)$$

which, for second order differential equations, means

$$\mathcal{L}[y(t)] = Y(s),\ \mathcal{L}[y'(t)] = sY(s) - y(0),\ \mathcal{L}[y''(t)] = s^2Y(s) - sy(0) - y'(0)$$

Second Order D.E. Solution:
(Assuming p(t) and q(t) are constants where p(t)=p and q(t)=q)

$$Y(s) = \frac{\mathcal{L}[g(t)] + (s+p)y(0) + y'(0)}{s^2 + ps + q}$$

$$y(t) = \mathcal{L}^{-1}[Y(s)] = \mathcal{L}^{-1}[\frac{\mathcal{L}[g(t)] + (s+p)y(0) + y'(0)}{s^2 + ps + q}]$$

Laplace Transform Table:
A table of common Laplace Transforms and their inverses is located in the textbook.

Laplace Transforms (Cont.)

Unit Step Functions and Impulse Functions:
Unit step functions and impulse functions are piece-wise inputs to a system modelled in a differential equation. They are defined:

$$u(t-c) = u_c = \begin{cases} 0 & t < c \\ 1 & c \leq t \end{cases}$$

$$\delta(t-c) = \delta_c = \begin{cases} 0 & t \neq c \\ 1 & t = c \end{cases}$$

Properties of Laplace Transform Revisited:
Given a Laplace Transform of a function multiplied by a unit step function or impulse function, the transform has the property:

$$\mathcal{L}[u_c f(t-c)] = e^{-cs}\mathcal{L}[f(t)] = e^{-cs}[F(s)]$$

$$\mathcal{L}[k\delta_c] = ke^{-cs}$$

Note that an impulse does not have a function multiplied by it, but can have a magnitude

Inverse Laplace Transform of Unit and Impulse Functions:

$$\mathcal{L}^{-1}[e^{-cs}f(s)] = u_c\mathcal{L}^{-1}[F(s)] = u_c f(t-c)$$