Vectors

Dot Product:
For any two vectors, $\vec{u}, \vec{v}$ that are dimension n, the dot product of the two vectors is defined:

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

This is further related to the angle between the two vectors by the formula:

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|}$$

Note: Any vector dotted with a perpendicular vector yields a result of 0.

Cross Product:
For any two vectors, $\vec{u}, \vec{v}$ that are dimension 3, the cross product of the two vectors is defined:

$$\vec{u} \times \vec{v} = det(\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}) =< u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1 >$$

The resultant of a cross product is orthogonal to both original vectors.
The magnitude of the resultant of the cross product is given by the formula:

$$|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin\theta$$

Note: Any vector crossed with a parallel vector yields a result of $\vec{0}$.

Parametric Representations:
For a given function $f(x, y)$ or $f(x, y, z)$, the curve can be parameterized by setting one variable equal to the parameter t and resolving the remaining terms as a function of the parameter.

Example: Parametrize the equation of a line $3x + 5y = 3$.

$$3x + 5y = 3 \rightarrow y(x) = \frac{3 - 3x}{5}$$

$$\text{Let } x = t \rightarrow y(t) = \frac{3 - 3t}{5}$$

$$\boxed{< x, y >=< t, \frac{3 - 3t}{5} >=< 0, \frac{3}{5} > +t < 1, -\frac{3}{5} >}$$

Note: For circles and ellipses, a common parameterization is $x = a\cos t, y = b\sin t$.

Planes:

Forms of Equations for Planes:
Planes can be represented in several ways:

$$\text{Vector Form: } \vec{n} < x, y, z >= \vec{n}P$$

$$\text{Scalar Form: } ax + by + cz = d$$

$$\text{Scalar Form: } a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

For these equations, $P =< x_0, y_0, z_0 >$ where P is a point on the plane and $\vec{n} =< a, b, c >$ where $\vec{n}$ is orthogonal to the plane.
Example:
Find the equation of a plane through the following three points:

$$P = (1, 0, -1), Q = (2, 2, 1), R = (4, 1, 2)$$

First, two co-planar vectors must be obtained from the three points:

$$\vec{PQ} =< 2, 2, 1 > - < 1, 0, -1 >=< 1, 2, 2 >$$

$$\vec{QR} =< 4, 1, 2 > - < 1, 0, -1 >=< 3, 1, 3 >$$

The cross product of the two vectors yields the normal vector:

$$\vec{PQ} \times \vec{QR} = det(\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 2 \\ 3 & 1 & 3 \end{bmatrix}) =< 4, 3, -5 >$$

Therefore the plane is:

$$\vec{n} < x, y, z >= \vec{n}P$$

$$\boxed{4x + 3y - 5z = 9}$$

Partial Derivatives:

General Comments:
For a function $f(x, y, z)$ the partial derivatives are defined formally as:

$$f_x(x, y, z) = \lim_{h \to 0} \frac{f(x+h, y, z) - f(x, y, z)}{h}$$

$$f_y(x, y, z) = \lim_{h \to 0} \frac{f(x, y+h, z) - f(x, y, z)}{h}$$

$$f_z(x, y, z) = \lim_{h \to 0} \frac{f(x, y, z+h) - f(x, y, z)}{h}$$

Informally, they can be considered the rate of change in a single direction while holding the remainder of the variables constant.

Gradients:
The gradient of a function is defined for a two and three dimensional function as:

$$\nabla f(x, y) =< f_x(x, y), f_y(x, y) >$$

$$\nabla f(x, y, z) =< f_x(x, y, z), f_y(x, y, z), f_y(x, y, z) >$$

The gradient has a chain rule property defined:

$$\frac{d}{dt} f(r(t)) = \nabla f(r(t)) \cdot r'(t)$$

Multivariate Optimization:

General Comments:
Much like Calc I optimization, minimums and maximums of functions lie along boundary conditions or at critical points. Whereas Calc I only had one variable (and thus one derivative set equal to zero), multivariate optimization is done on functions with multiple variable with multiple derivatives.

They are solved then, by solving:

$$\nabla f(x, y, z) = \vec{0}$$

An equation similar to the second derivative test can be used to test for minimums, maximums, and saddle points in $\mathbb{R}^3$.

$$D = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - f_{xy}(x_0, y_0)^2$$

The possible results are tabulated below:

Result	**Classification**
$D > 0$ and $f_{xx} < 0$	Relative Maximum
$D > 0$ and $f_{xx} > 0$	Relative Minimum
$D < 0$	Saddle Point

Lagrange Multiplier:
When optimizing one function along the curve of another, a common method of solution is the Lagrange Multiplier.

Given f(x,y,z) and g(x,y,z) = 0

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$

where λ is an unknown constant.

Solving all partial derivatives for λ yields:

$$\lambda = \frac{\partial f / \partial x}{\partial g / \partial x} = \frac{\partial f / \partial y}{\partial g / \partial y} = \frac{\partial f / \partial z}{\partial g / \partial z}$$

This allows for substitutions to be solved for which can be inserted into g(x,y,z) to obtain solution points.

A final value is obtained by plugging the solution points (and any boundary conditions) into f(x,y,z).

Multivariate Integration:

General Comments:
Double and triple intregrals are used to obtain the area and volume respectively as bounded by a set of conditions. They are expressed:

$$A = \int \int f(x,y)dA$$

$$V = \int \int \int f(x,y,z)dV$$

Application:
Similar to Calc II, double and triple integrals can be used to determine the centroid or center of mass of a region. Such coordinates are given by:

$$\bar{c} = (\bar{x}, \bar{y}) = (\frac{\int \int xdA}{\int \int dA}, \frac{\int \int ydA}{\int \int dA})$$

$$\bar{c_m} = (\bar{x}, \bar{y}, , \bar{z}) = (\frac{\int \int \int xdV}{\int \int \int dV}, \frac{\int \int \int ydV}{\int \int \int dV}, \frac{\int \int \int zdV}{\int \int \int dV})$$

Note: Some areas/volumes are easily obtained through geometry (e.g $A_{circle} = \pi r^2$)

Coordinate Systems:
When converting between rectangular and cylindrical or spherical coordinates, the following conversions are made:

Rectangular	**Cylindrical**	**Spherical**
x	$r\cos\theta$	$\rho\sin\theta\sin\Phi$
y	$r\sin\theta$	$\rho\sin\theta\cos\Phi$
z	z	$\rho\cos\Phi$
x^2+y^2	r^2	$\rho^2\sin^2\Phi$
$x^2+y^2+z^2$	r^2+z^2	ρ^2
y/x	$\tan\theta$	$\tan\theta$

Coordinates	**dA**	**dV**
Rectangular	$dxdy$	$dxdydz$
Cylindrical	$rdrd\theta$	$rdrd\theta dz$
Spherical	—	$(\rho^2\sin\Phi)d\theta d\Phi$