Integration Guide

General Comments:
Given the integral,

$$F(x) = \int f(x)dx$$

$F(x)$ can be obtained using multiple techniques dependent on the form of $f(x)$.

Substitution ("U-Sub"):
If $f(x)$ is of the form $f(x) = A * g'(x) * g(u(x))$ then the method of substitution may be used to solve the integral. In general, this is determined by observation. The process is completed as follows:

$$F(x) = \int f(x)dx = \int Ah(g(x))g'(x)dx = A\int h(g(x))g'(x)dx$$

Let $u = g(x)$ and note $\frac{du}{dx} = g'(x) \rightarrow g'(x)dx = du$. Reorganize and make this substitution:

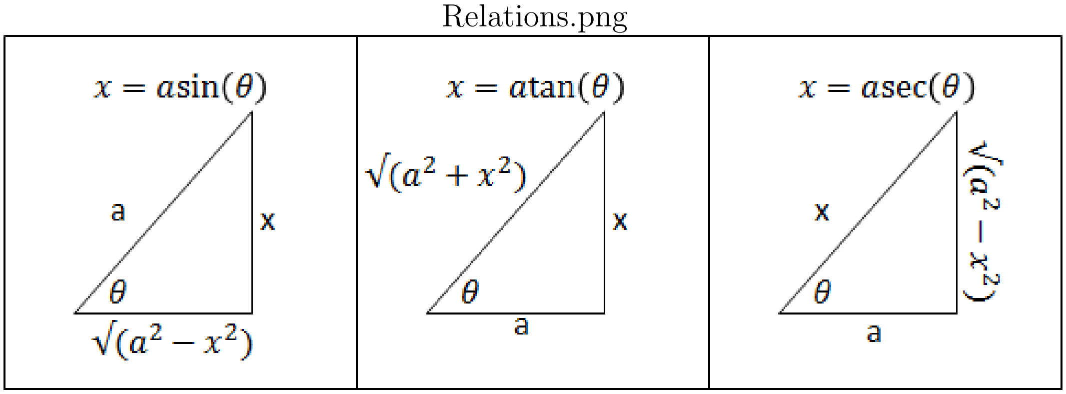
$$F(x) = A\int h(g(x))g'(x)dx = A\int h(u)du = A(H(u) + C)$$

$$F(x) = AH(g(x)) + C$$

Trigonometric Substitution:
If $f(x)$ is comprised of one of the following forms, the corresponding trig sub should be used and the integral completed through the above substitution method:

Term in f(x)	**x Substitution**	**f(x) Substitution**
$\sqrt{a^2 - x^2}$	$x = a\sin(\theta)$	$\sqrt{a^2 - x^2} = \cos(\theta)$
$\sqrt{a^2 + x^2}$	$x = a\tan(\theta)$	$\sqrt{a^2 + x^2} = \sec(\theta)$
$\sqrt{x^2 - a^2}$	$x = a\sec(\theta)$	$\sqrt{x^2 - a^2} = \tan(\theta)$

Relations.png



Integration Guide

Integration by Parts:
If $f(x)$ is the product of two functions ($f(x) = (g * h)(x)$) then the method of integration by parts can be used. Integration by parts is completed by grouping an integrable function (h(x)) with the differential term (dx) such that $h(x)dx = dv$. The remaining $g(x)$ function in renamed $u(x)$ for consistency with the book:

$$F(x) = \int f(x)dx = \int (g * h)(x)dx = \int g(x) * (h(x)dx) = \int u(x)dv$$

Integration by parts then follows the formula:

$$F(x) = \int u(x)dv = u * v - \int v * du$$

Partial Fractions:
If $f(x)$ is comprised of two polynomials being divided $\frac{P}{Q}$, the method of partial fractions can be used.
First, the denominator of n^{th} degree is factored into its simplest linear and quadratic components:

$$Q = (x - q_1)(x - q_2)\dots(x - q_n) \rightarrow \frac{P}{Q} = \frac{P}{(x - q_1)(x - q_2)\dots(x - q_n)}$$

This fraction can then be separated into additive terms over each of the denominator's components:

$$\frac{P}{Q} = \frac{P}{(x - q_1)(x - q_2)\dots(x - q_n)} = \frac{A_1}{(x - q_1)} + \frac{A_2}{(x - q_2)} + \dots + \frac{A_n}{(x - q_n)}$$

Each of these terms can be integrated independently.

Partial Fractions SPECIAL CASES:
If one of the factors of Q cannot be reduced to the form $(x - q)$ or is repeated, the additive term changes form according to the rules below:

If a factor of Q cannot be reduced to a binomial, then the factor will be quadratic ($ax^2 + bx + c$). Its additive term has the form:

$$\text{Quadratic Factor Case: } Term = \frac{Ax + B}{ax^2 + bx + c}$$

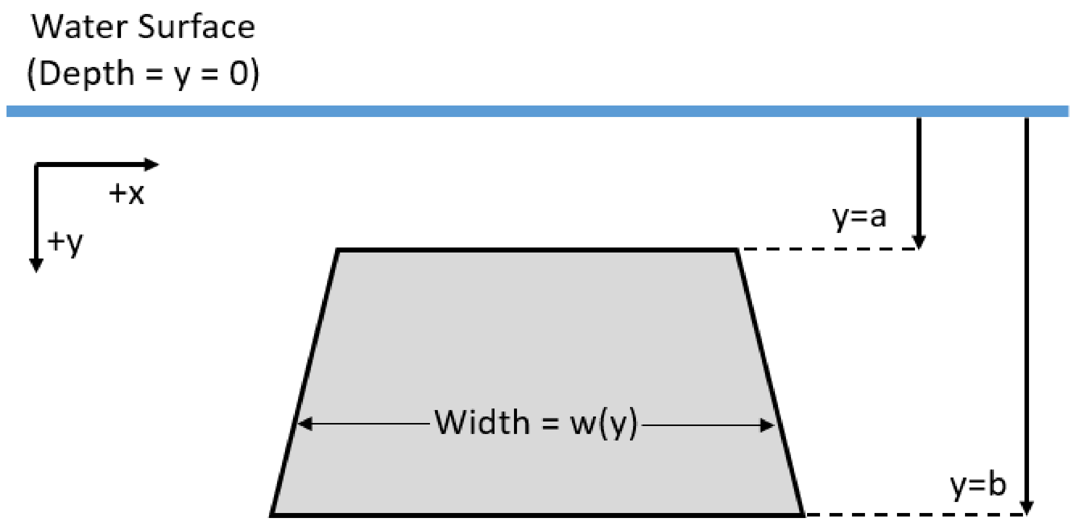
If a factor of Q is repeated, then there will be a series of additive factors corresponding to different powers of the factor. For a factor repeated n times, the following term would be used:

$$\text{Repeated Factor Case: } Term = \frac{A_1}{(x - q)} + \frac{A_2 x}{(x - q)^2} + \dots + \frac{A_n x^{n-1}}{(x - q)^n}$$

Fluid Force

General Overview:
Given a fluid force problem similar to the image below, the solution is as follows:

Force Diagram.png



Brief Definitions and Fluid Dynamics Review:
Fluid force on an object is defined by the hydrostatics equation:

$$F = \int_a^b (\rho g W(y) y) dy$$

where the variables are defined:

Variable	Metric	English
ρ	Fluid Density ($\frac{kg}{m^3}$)	Fluid Density ($\frac{lbm}{ft^3}$)
g	Gravitational Acceleration $9.8\frac{m}{s^2}$	Gravitational Acceleration $32.2\frac{ft}{s^2}$
ρg	(See Above)	(For water) $62.4\ \frac{lbf}{ft^3}$
W(y)	Cross-Sectional Width at Depth y (m)	Cross-Sectional Width at Depth y (ft)
y	Depth (m)	Depth (ft)
dy	Differential Depth (m)	Differential Depth (ft)

Fluid Force

Solution Method:
Overall, there are generally three main steps to solving a fluid force problem:

1. Determine the function $W(y)$ for the submerged object for all depths on the interval $y \in [a, b]$.
   (a) If no continuous $W(y)$ exists for $y \in [a, b]$, then divide the interval into sub-intervals such that $W_1(y)$ is continuous on $[a, y_1]$, $W_2(y)$ is continuous on $[y_1, y_2]$, ..., $W_n(y)$ is continuous on $[y_{n-1}, b]$
   (b) Break up the hydrostatics equation over the new intervals:

$$F = \int_a^{y_1} (\rho g W_1(y) y) dy + \int_{y_1}^{y_2} (\rho g W_2(y) y) dy + \cdots + \int_{y_{n-1}}^{b} (\rho g W_n(y) y) dy$$

2. Plug in knowns and $W(y)$ to hysdrostatics equation above.
3. Solve integral. (See integration methods sheet as needed)

Example:
Take the above picture with the following values: $a = 1\text{m}, b = 2\text{m}, \rho = 10\frac{\text{kg}}{\text{m}^3}$ and assume the shape has base widths of $W(a) = 1\text{m}$ and $W(b) = 2\text{m}$.

1. Determine $W(y)$: The width of the trapezoid in the image is bounded by the left and right lines. As an adaptation of the point-slope form of a line, an equation for the width can be established as:

$$W(y) = W(a) + \frac{\Delta W}{\Delta y}(y - a) = W(a) + ((\frac{\Delta W}{\Delta y})_{LHS} + (\frac{\Delta W}{\Delta y})_{RHS})(y - a)$$

$$(\frac{\Delta W}{\Delta y})_{LHS} = \frac{1/2\text{m}}{1\text{m}} = (\frac{\Delta W}{\Delta y})_{RHS}$$

$$W(y) = 1 + (y - 1)$$

2. Plug in knowns to hydrostatic equation:

$$F = \int_a^b (\rho g W(y) y) dy = \int_{1\text{m}}^{2\text{m}} (10\frac{\text{kg}}{\text{m}^3})(9.8\frac{\text{m}}{\text{s}^2})(1 + (y - 1)) y) dy$$

3. Solve:

$$F = 98\frac{\text{N}}{\text{m}^3} \int_{1\text{m}}^{2\text{m}} (1 + (y - 1)) y) dy = 98\frac{\text{N}}{\text{m}^3} \int_{1\text{m}}^{2\text{m}} y^2 dy = 98\frac{\text{N}}{\text{m}^3} (\frac{y^3}{3}|_{1\text{m}}^{2\text{m}})$$

$$F = 98\frac{\text{N}}{\text{m}^3} (\frac{7}{3}\text{m}^3) = 228.7N$$

Numerical Approximations

Integrals:
In general, the numerical integration methods below are more accurate as you proceed lower into the list.
Over an interval [a,b] the integral of a function can be approximated by dividing the interval into N sections:

1. <u>*Left Side and Right Side Methods*</u>

$$\int_a^b f(x)dx = (\frac{b-a}{N})\sum_{i=k}^{N} f(a+\frac{b-a}{N}i)$$

where k=0 for Left Side approximations and k = 1 for Right Side approximations

2. <u>*Trapezoidal Method*</u>

$$\int_a^b f(x)dx = (\frac{b-a}{N})\sum_{i=0}^{N}[(\frac{1.5+.5(-1)^i}{2})f(a+\frac{b-a}{N}i)]$$

$$= (\frac{b-a}{2N})(f(a)+2f(a+\frac{b-a}{N})+f(a+2\frac{b-a}{N})+\cdots+2f(a+(N-1)\frac{b-a}{N})+f(b))$$

$$= \frac{1}{2}(R_N+L_N)$$

Where R_N and L_N are Right and Left Side approximations.

3. <u>*Simpson's Method*</u>

$$\text{Let } \Delta x = \frac{b-a}{N}$$

$$\int_a^b f(x)dx = (\frac{\Delta x}{3})(f(a)+4f(a+\Delta x)+2f(a+2\Delta x)+\cdots+4f(a+(N-1)\Delta x)+f(b))$$

where the coefficients are 1 for the first and last term and follow the pattern of alternating 4 and 2 for all other terms.

Note: N MUST be even for Simpsons Method

Method	**Error** $\leq$
Midpoint Method	$\frac{K_2(b-a)^3}{24N^2}$
Trapezoidal Method	$\frac{K_2(b-a)^3}{12N^2}$
Simpsons Method	$\frac{K_4(b-a)^5}{180N^4}$

Where K_i is the max of the i_{th} derivative over the interval $[a,b]$

Numerical Approximations

Summations:
Given an alternating and converging series, an approximation of the infinite summation (S_N) can be made with an error bound given by:

$$S = \sum_{n=1}^{\infty} a_n, S_N = \sum_{n=1}^{N} a_n$$

$$|S - S_N| = Error \leq |a_{N+1}|$$

Taylor Series:
A function $f(x)$ can be approximated as a finite Taylor series centered at a single point $x = c$ by the equation:

$$f(x) \approx T_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(c)}{n!}(x - c)^n$$

An N^{th} degree Taylor Polynomial has an error bound for approximating any $x = a$ given by:

$$Error \leq \frac{K_{N+1}}{(N+1)!}(x - c)^{N+1}$$

where K_{N+1} is the maximum absolute value of the f^{N+1} on the interval $[a, c]$ (or $[c, a]$).

See Section 10.7 for a list of common Taylor Series