General Comments: 
Given the differential equation,
\[ y' + p(t)y = g(t) \]
the solution takes the form:
\[ y(t) = c_1 y_1 \]
where \( y_1 \) is obtained from using the integrating factor method.

Derivation:
Define the integrating factor as:
\[ \mu = e^{\int p(t) dt} \]
Observe that taking the derivative with respect to \( t \) yields:
\[ \mu' = \frac{d}{dt}( \int p(t) dt ) e^{\int p(t) dt} = p(t) e^{\int p(t) dt} = p(t) \mu \]
Distributing the integrating factor through the differential equation reveals an internal product rule on the left hand side of the equation.
\[ \mu(y' + p(t)y) = \mu g(t) \]
\[ \mu y' + (\mu p(t))y = \mu y' + \mu' y = \mu g(t) \]
\[ (y\mu)' = \mu g(t) \]
Integrating the sides with respect to \( t \) and dividing by the integrating factor solves the differential equation
\[ \int (y\mu)' dt = \int \mu g(t) dt \]
\[ y\mu = \int \mu g(t) dt \]
\[ y = \frac{1}{\mu} \int \mu g(t) dt \]

Solution Steps:
\[ y' + p(t)y = g(t) \]
\[ \mu = e^{\int p(t) dt} \]
\[ y = \frac{1}{\mu} \int \mu g(t) dt \]
Exact Equations

General Comments:
Given the differential equation,
\[ M(x, y) + N(x, y) \frac{dy}{dx} = 0 \]
the method of exact equations can be used to solve for \( y(x) \)

Testing Exactness:
If the function is exact in its current state, then
\[ \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \text{ and } \frac{d\mu}{dx} = 0 \]
otherswise an integrating factor \( \mu \neq 0 \) is required. This integrating factor can take two forms depending on assumption that it is purely a function of \( x \) or \( y \):
\[ \frac{d\mu}{dx} = \frac{M_y - N_x}{N} \mu = -\frac{M_y - N_x}{M} \mu \]
which yields
\[ \frac{\partial \mu M}{\partial y} = \frac{\partial \mu N}{\partial x} \]

Solution Method:
\[ \mu (M(x, y) + N(x, y) \frac{dy}{dx}) = \mu M(x, y)dx + \mu N(x, y)dy = 0 \]
\[ \Psi(x, y) = \int (\mu M(x, y)dx + \mu N(x, y)dy) = \int 0dx = c \]
\[ \Psi(x, y) = \int \mu M(x, y)dx + \int \mu N(x, y)dy = c \]

Removing Redundant Terms:
The resulting solution equation \( \Psi \) must be such that, omitting constants \( C \),
\[ \Psi(x, y) = \int \mu M(x, y)dx = \int \mu N(x, y)dy \]
which allows for the reduction of redundant terms.
Tank Mixture Problems

General Comments:
Given a tank problem drawn below where $x_i(t)$ is the amount of substance mixed with water, $V_i$ is the volume of the tank, the solution is as follows:

Flow Diagram.png

General Differential Equation:
For the tank system, a differential equation for each tank is written as the rate of mixture in minus the rate of mixture out. Note the concentrations ($\gamma$) and flow rate ($\dot{V}$) of each stream leaving and entering the tank.

\[
\frac{d(x_i)}{dt} = \Sigma(\gamma \dot{V})_{in} - \Sigma(\gamma \dot{V})_{out}
\]

Note that fluid leaving a "well stirred tank" has concentration $\gamma_{well-stirred} = \frac{x_i(t)}{V_i}$

Solution Method: Summing all the concentrations and flow rates in and out of tank 1 and two yields the following system of equations:

\[
\frac{d(x_1)}{dt} = \gamma_{in} \dot{V}_{in} + \gamma_{21} \dot{V}_{21} - \gamma_{out,1} \dot{V}_{out,1} - \gamma_{12} \dot{V}_{12}
\]
\[
\frac{d(x_2)}{dt} = \gamma_{12} \dot{V}_{12} - \gamma_{21} \dot{V}_{21} - \gamma_{out,2} \dot{V}_{out,2}
\]

Replacing the concentration terms with their well-stirred terms:

\[
\frac{d(x_1)}{dt} = \gamma_{in} \dot{V}_{in} + \frac{x_2}{V_2} \dot{V}_{21} - \frac{x_1}{V_1} \dot{V}_{out,1} - \frac{x_1}{V_1} \dot{V}_{12} = \gamma_{in} \dot{V}_{in} + \frac{x_2}{V_2} \dot{V}_{21} - \frac{x_1}{V_1} (\dot{V}_{out,1} + \dot{V}_{12})
\]
\[
\frac{d(x_2)}{dt} = \frac{x_1}{V_1} \dot{V}_{12} - \frac{x_2}{V_2} \dot{V}_{21} - \frac{x_2}{V_2} \dot{V}_{out,2} = \frac{x_1}{V_1} \dot{V}_{12} - \frac{x_2}{V_2} (\dot{V}_{out,2} + \dot{V}_{21})
\]

Note: If the tank system has only one tank, only the first equation is required with $\dot{V}_{21} = \dot{V}_{12} = 0$
Elimination Method:
Similar to how linear systems of equations were solved in College Algebra, adding and subtracting the differential equations here will be used to reduce the equation to a function of only one variable.

First each equation is set equal to 0:

\[
\frac{d(x_1)}{dt} = \gamma_{in}\dot{V}_{in} + x_2 \frac{\dot{V}_{21}}{V_2} - x_1 \left( \frac{\dot{V}_{out,1} + \dot{V}_{12}}{V_1} \right) \rightarrow \frac{d(x_1)}{dt} - \gamma_{in}\dot{V}_{in} - x_2 \frac{\dot{V}_{21}}{V_2} + x_1 \left( \frac{\dot{V}_{out,1} + \dot{V}_{12}}{V_1} \right) = 0
\]

\[
\frac{d(x_2)}{dt} = x_1 \frac{\dot{V}_{12}}{V_1} - x_2 \left( \frac{\dot{V}_{out,2} + \dot{V}_{21}}{V_2} \right) \rightarrow \frac{d(x_2)}{dt} - x_1 \frac{\dot{V}_{12}}{V_1} + x_2 \left( \frac{\dot{V}_{out,2} + \dot{V}_{21}}{V_2} \right) = 0
\]

Let \( D = \frac{d}{dt} \) such that \( D[x_i] = \frac{dx_i}{dt} \) and factor out the \( x_1, x_2 \) terms as follows:

\[
(D + \left( \frac{\dot{V}_{out,1} + \dot{V}_{12}}{V_1} \right)) [x_1] - \gamma_{in}\dot{V}_{in} - x_2 \frac{\dot{V}_{21}}{V_2} = 0
\]

\[
(D + \left( \frac{\dot{V}_{out,2} + \dot{V}_{21}}{V_2} \right)) [x_2] - x_1 \frac{\dot{V}_{12}}{V_1} = 0
\]

Multiplying equation one by \( \frac{\dot{V}_{12}}{V_1} \) and equation two by \( (D + \left( \frac{\dot{V}_{out,1} + \dot{V}_{12}}{V_1} \right)) \) allows us to cancel the \( x_1 \) terms through addition of the two equation and results in the differential equation:

\[
(D + \left( \frac{\dot{V}_{out,1} + \dot{V}_{12}}{V_1} \right)) (D + \left( \frac{\dot{V}_{out,2} + \dot{V}_{21}}{V_2} \right)) [x_2] - x_2 \frac{\dot{V}_{21}}{V_2} = 0
\]

\[
\left( D^2 + \left( \frac{\dot{V}_{out,1} + \dot{V}_{12}}{V_1} + \frac{\dot{V}_{out,2} + \dot{V}_{21}}{V_2} \right) D + \left( \frac{\dot{V}_{out,1} + \dot{V}_{12}}{V_1} \right) \left( \frac{\dot{V}_{out,2} + \dot{V}_{21}}{V_2} \right) \right) [x_2] - \frac{\dot{V}_{12}}{V_1} (\gamma_{in}\dot{V}_{in} + x_2 \frac{\dot{V}_{21}}{V_2}) = 0
\]

\[
\frac{d^2(x_2)}{dt^2} + \left( \frac{\dot{V}_{out,1} + \dot{V}_{12}}{V_1} + \frac{\dot{V}_{out,2} + \dot{V}_{21}}{V_2} \right) \frac{d(x_2)}{dt} + \left( \frac{\dot{V}_{out,1} + \dot{V}_{12}}{V_1} \right) \left( \frac{\dot{V}_{out,2} + \dot{V}_{21}}{V_2} \right) x_2 = \frac{\dot{V}_{12}}{V_1} \gamma_{in}\dot{V}_{in}
\]

\( x_2 \) can be solved by use of the following methods of solving second order non-homogeneous differential equations.

Further, \( x_1 \) can be solved for by another elimination method or by substituting \( x_2, x'_2 \) back into the above differential equation.
Homogeneous Equations

General Comments:
Given the differential equation,
\[
\frac{d^2y}{dt^2} + p(t) \frac{dy}{dt} + q(t)y = 0
\]
the solution takes the form:
\[
y(t) = c_1y_1 + c_2y_2
\]
where \( y_1 \) and \( y_2 \) are obtained by solving the characteristic equation of the differential equation.

Characteristic Equation:
If \( p(t) \) and \( q(t) \) are constants, assume a solution of \( y = e^{\lambda t} \) with first and second derivatives \( \lambda e^{\lambda t} \) and \( \lambda^2 e^{\lambda t} \) respectively.

Plugging this into the differential equation yields:
\[
e^{\lambda t} (\lambda^2 + p(t)\lambda + q(t)) = 0
\]
which is solved through the quadratic formula to yield homogeneous solutions \( y_1 \) and \( y_2 \).

\[
\lambda_{1,2} = \frac{-p(t) \pm \sqrt{p(t)^2 - 4q(t)}}{2} \rightarrow y_1 = e^{\lambda_1 t}, \: y_2 = e^{\lambda_2 t}
\]

Repeated Roots:
If \( \lambda_1 = \lambda_2 \) then the repeated homogeneous solution is multiplied by \( t \) such that
\[
y_1 = e^{\lambda t}, \: y_2 = te^{\lambda t}
\]

Complex Roots:
If \( p(t)^2 - 4q(t) < 0 \) such that the roots of the characteristic equation are complex, they take the form:
\[
\lambda_{1,2} = \frac{-p(t)}{2} \pm \frac{\sqrt{4q(t) - p(t)^2}}{2} i = \eta \pm \omega i \rightarrow y_1 = e^{(\eta + \omega)i t}, \: y_2 = e^{(\eta - \omega)i t}
\]
by the Euler identity \( e^{i\alpha} = \cos(\alpha) + i\sin(\alpha) \) and rearranging of the variables, the solution becomes
\[
y_1 = e^{\eta t}\cos(\omega t), \: y_2 = e^{\eta t}\sin(\omega t)
\]
Nonhomogeneous Equations - Method of Undetermined Coefficients

**General Comments:**
Given the differential equation,

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = g(t)$$

the solution takes the form:

$$y(t) = c_1y_1 + c_2y_2 + y_p$$

where $y_1$ and $y_2$ are obtained by solving the differential equation as a homogeneous equation and method of undetermined coefficients is utilized to obtain $y_p$.

For the method of undetermined coefficients, $y_p$ is defined based off of $g(t)$ and the solutions to the homogeneous case.

**Choosing $y_p$'s Form from $g(t)$:**
- If $g(t)$ has a term $t^n$, then $y_p$ should be assumed to contain a polynomial of degree $n$ such that $y_p = (A_nt^n + A_{n-1}t^{n-1} + \ldots + A_0) + h(t)$ where $h(t)$ is any other functions from this section.

- If $g(t)$ has a term $\sin(\omega t)$ or $\cos(\omega t)$, then $y_p$ should be assumed as $y_p = (A_0\cos(\omega t) + A_1\sin(\omega t)) + h(t)$ where $h(t)$ is any other functions from this section.

- If $g(t)$ has a term $e^{\alpha t}$, then $y_p$ should be assumed as $y_p = A_0e^{\alpha t} + h(t)$ where $h(t)$ is any other functions from this section.

**Choosing $y_p$'s Form from Homogeneous Solutions:**
- If the proposed solution created from the above $g(t)$ rules has a term that matches the form of any solutions to the homogeneous equation, the anticipated solution should be multiplied by $t$.

**Solution of Coefficients:**
Take all required derivatives of the proposed solution and plug in to the differential equation:

$$y''_p + p(t)y'_p + q(t)y_p = g(t)$$

and solve for all constants in $y_p$. 
Nonhomogeneous Equations - Variation of Parameters

General Comments:
Given the differential equation,
\[ \frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = g(t) \]
the solution takes the form:
\[ y(t) = c_1y_1 + c_2y_2 + y_p \]
where \( y_1 \) and \( y_2 \) are obtained by solving the differential equation as a homogeneous equation and variation of parameters is utilized to obtain \( y_p \).

Particular Solution:
For variation of parameters, \( y_p \) is defined:
\[ y_p = \nu_1 y_1 + \nu_2 y_2, \quad \nu_1 = \int \frac{-g(t)y_2}{W} dt, \quad \text{and} \quad \nu_2 = \int \frac{g(t)y_1}{W} dt \]
\[ \text{Note that } \nu_1 \text{ includes a negative sign but } \nu_2 \text{ does not.} \]
with \( W \) in this case being the Wronskian.

The Wronskian can be calculated by the following formula:
\[ W = \det \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = y_1y_2' - y_2y_1' \]
\[ \text{Note that the Wronskian cannot be zero as used in the integration above.} \]

Reduction of Order:
If \( y_1(t) \) is not identically 0, then the homogeneous solutions to the differential equation are related by
\[ y_2 = y_1 \int e^{-\int p(t)dt} \frac{y_2}{y_1^2} dt \]
Laplace Transforms

General Comments:
Given the differential equation and initial conditions,
\[
\frac{d^2 y}{dt^2} + p(t) \frac{dy}{dt} + q(t) y = g(t), \quad y(0) = k_1, \quad y'(0) = k_2
\]
the solution takes the form:
\[
y(t) = c_1 y_1 + c_2 y_2 + y_p
\]
where \( y_1, \ y_2, \) and \( y_p \) are obtained by transforming the differential into the Laplace domain and returning the solution to the time domain.

Properties of the Laplace Transform:
- The Laplace Transform is a linear operator which means it can be "distributed" over terms being added or subtracted. I.e.
  \[
  \mathcal{L}[A(t) + B(t)] = \mathcal{L}[A(t)] + \mathcal{L}[B(t)]
  \]
- Scalar multiples of functions undergoing Laplace Tranforms are as follows:
  \[
  \mathcal{L}[cA(t)] = c\mathcal{L}[A(t)]
  \]

Laplace Transform of \( y^{(n)}(t) \):
The Laplace Transform of any derivative of the solution function yields,
\[
\mathcal{L}[y^{(n)}(t)] = s^nY(s) - s^{n-1}y(0) - s^{n-2}y'(0) - \ldots - y^{n-1}(0)
\]
which, for second order differential equations, means
\[
\mathcal{L}[y(t)] = Y(s), \quad \mathcal{L}[y'(t)] = sY(s) - y(0), \quad \mathcal{L}[y''(t)] = s^2Y(s) - sy(0) - y'(0)
\]

Second Order D.E. Solution:
(Assuming \( p(t) \) and \( q(t) \) are constants where \( p(t)=p \) and \( q(t)=q \))
\[
Y(s) = \frac{\mathcal{L}[g(t)] + (s + p)y(0) + y'(0)}{s^2 + ps + q}
\]
\[
y(t) = \mathcal{L}^{-1}[Y(s)] = \mathcal{L}^{-1}\left[\frac{\mathcal{L}[g(t)] + (s + p)y(0) + y'(0)}{s^2 + ps + q}\right]
\]

Laplace Transform Table:
A table of common Laplace Transforms and their inverses is located in the textbook.
Unit Step Functions and Impulse Functions:
Unit step functions and impulse functions are piece-wise inputs to a system modelled in a differential equation. They are defined:

\[ u(t - c) = u_c = \begin{cases} 
0 & t < c \\
1 & c \leq t
\end{cases} \]

\[ \delta(t - c) = \delta_c = \begin{cases} 
0 & t \neq c \\
1 & t = c
\end{cases} \]

Properties of Laplace Transform Revisited:
Given a Laplace Transform of a function multiplied by a unit step function or impulse function, the transform has the property:

\[ \mathcal{L}[u_c f(t - c)] = e^{-cs} \mathcal{L}[f(t)] = e^{-cs} [F(s)] \]

\[ \mathcal{L}[k \delta_c] = ke^{-cs} \]

Note that an impulse does not have a function multiplied by it, but can have a magnitude

Inverse Laplace Transform of Unit and Impulse Functions:

\[ \mathcal{L}^{-1}[e^{-cs} f(s)] = u_c \mathcal{L}^{-1}[F(s)] = u_c f(t - c) \]