REVIEW FOR POST-EXAM III MATERIAL

This review is about the material covered after Exam III, namely the parts of section 5.5 on quadratic forms and graphing quadratic equations.

QUADRATIC FORMS

1. A quadratic form in two variables is a polynomial of the form

$$q(x,y) = ax^2 + by^2 + cxy$$

We can express this using matrix multiplication as follows. Let $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $A = \begin{bmatrix} a & c/2 \\ c/2 & b \end{bmatrix}$. Then we have

$$q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & c/2 \\ c/2 & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = ax^2 + by^2 + cxy = q(x,y)$$

2. A quadratic form in three variables is a polynomial of the form

$$q(x,y,z) = ax^2 + by^2 + cz^2 + dxy + exz + fyz$$

We can also express this using matrix multiplication by letting $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $A = \begin{bmatrix} a & d/2 & e/2 \\ d/2 & b & f/2 \\ e/2 & f/2 & c \end{bmatrix}$. Then as before we have $q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} = q(x,y,z)$.

3. A quadratic form in n variables $x_1, \ldots, x_n$ can be most easily defined as $q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} = q(x_1, \ldots, x_n)$, where $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ and A is a symmetric matrix.

4. The Principal Axes Theorem says that every quadratic form can be diagonalized, i.e. we can make a change of variables from $x_1, \ldots, x_n$ to $x'_1, \ldots, x'_n$ so that $q(x_1, \ldots, x_n) = d_1(x'_1)^2 + \cdots + d_n(x'_n)^2$. This is equivalent to saying that we can change variables so that our new matrix A' is diagonal.

   The way this works is as follows: Since A is symmetric it is orthogonally diagonalizable, so $Q^T AQ = D$ for some orthogonal matrix Q and diagonal matrix D. We set $\mathbf{x} = Q\mathbf{x}'$. Then $(\mathbf{x}')^T Q^T AQ\mathbf{x}' = (\mathbf{x}')^T D\mathbf{x}'$.

   Note that the diagonal entries of D are the eigenvalues of A.

5. A quadratic form is positive definite if for every $\mathbf{x} \neq \mathbf{0}$ we have that $q(\mathbf{x}) > 0$; since $q(\mathbf{x}) = d_1(x_1')^2 + \cdots + d_n(x_n')^2$ this happens if and only if all the eigenvalues are positive.

   Similarly a quadratic form is negative definite if $q(\mathbf{x}) < 0$ for every non-zero $\mathbf{x}$, and this is equivalent to all the eigenvalues being negative.

   If one replaces the $>$ and $<$ signs in these definitions by $\geq$ and $\leq$ one gets the definitions of positive semidefinite and negative semidefinite; these are equivalent to having all the eigevalues $\lambda \geq 0$ or $\lambda \leq 0$, respectively.

   If there are non-zero vectors $\mathbf{x}$ and $\mathbf{y}$ such that $q(\mathbf{x}) > 0$ and $q(\mathbf{y}) < 0$, then the form is indefinite. This corresponds to having both positive and negative eigenvalues.

6. As an example, let $q(x, y, z) = (9/2)x^2 + (9/2)y^2 + 3z^2 - 3xy$. This has matrix $A = \begin{bmatrix} 9/2 & -3/2 & 0 \\ -3/2 & 9/2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. The matrices Q and D in $Q^TAQ = D$ turn out to be those from the previous section

$$Q = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0 \end{bmatrix} \quad D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

   Thus, when we change coordinates using $\mathbf{x} = Q\mathbf{x}'$ we get that the form equals $3(x')^2 + 3(y')^2 + 9(z')^2$, and so it is positive definite.

GRAPHING QUADRATIC EQUATIONS

1. A quadratic equation in two variables is an equation of the form

$$ax^2 + by^2 + cxy + dx + ey + f = 0$$

   To graph this equation in the x-y plane we generally need to change variables to get something more recognizable, like the equation of a conic section such as an ellipse or hyperbola. There are generally two steps involved: a rotation (or reflection) which removes terms of the form cxy and a translation which removes terms of the form dx or ey. Before doing either of these we need to rewrite the equation in the matrix form

$$\mathbf{x}^T A\mathbf{x} + B\mathbf{x} + f = 0$$

   where $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, $A = \begin{bmatrix} a & c/2 \\ c/2 & b \end{bmatrix}$, and $B = \begin{bmatrix} d & e \end{bmatrix}$.

2. To eliminate the cxy type term we diagonalize the quadratic form ax^2+by^2+cxy by orthogonally diagonalizing the matrix A. We set $\mathbf{x} = Q\mathbf{x}'$ and substitute this into our equation to get

$$(\mathbf{x}')^T D\mathbf{x}' + C\mathbf{x}' + f = 0$$

where $C = BQ = \begin{bmatrix} g & h \end{bmatrix}$. This new equation has the form

$$d_1(x')^2 + d_2(y')^2 + gx' + hy' + f = 0$$

3. The next step would be to rewrite this equation in the form

$$d_1(x' - r)^2 + d_2(y' - s)^2 = k$$

by completing the square. For example if we had

$$25(x')^2 + 4(y')^2 - 100x' - 24y' + 36 = 0$$

then we would reorganize as follows.

$$25(x')^2 - 100x' + 4(y')^2 - 24y' + 36 = 0$$

$$25[(x')^2 - 4x'] + 4[(y')^2 - 6y] + 36 = 0$$

$$25[(x')^2 - 4x' + 4 - 4] + 4[(y')^2 - 6y + 9 - 9] + 36 = 0$$

$$25[(x' - 2)^2 - 4] + 4[(y' - 3)^2 - 9] + 36 = 0$$

$$25(x' - 2)^2 - 100 + 4(y' - 3)^2 - 36 + 36 = 0$$

$$25(x' - 2)^2 + 4(y' - 3)^2 = 100$$

Dividing by 100 and setting $x'' = x' - 2$ and $y'' = y' - 3$ gives the equation of an ellipse.

$$\frac{(x'')^2}{2^2} + \frac{(y'')^2}{5^2} = 1$$

Note that if the coefficient of $(x')^2$ or $(y')^2$ had been zero we would not have been able to complete the square for that variable and so would not have been able to eliminate the x' or y' term. In this case we might have gotten a parabola or a degenerate conic such as a line.

4. Obviously this can be done for $n > 2$ as well. For $n = 3$ the graphs are called quadric surfaces. See the class notes as well as page 438 in the book for the graphs and equations of these surfaces.

5. Note that if all you are asked to do is to identify the conic or quadric, then all you need to know is the set of signs of the eigenvalues of the matrix and the sign of the right hand side of the equation. For example, suppose you have a quadric $q(x, y) = ax^2 + bxy + cy^2 = d$ in $\mathbb{R}^2$, where $d > 0$. Let $A = \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix}$. If the eigenvalues of A are both positive, then you have an ellipse. If one is positive and the other negative, then you have a hyperbola. The general situation is similar both more complicated.

OPTIONAL TOPIC: CONGRUENCE OF SYMMETRIC MATRICES

Finding eigenvalues is hard. If all you want to do is to find the signs of the eigenvalues, then there is a very easy way to do this. Every symmetric matrix A is *congruent* to a diagonal matrix D. What this means is that there is a non-singular matrix P such that $P^T AP = D$. Here P need not be orthogonal, so P^T might not be equal to P^{-1}, so these matrices are not necessarily similar. The diagonal entries of D need not be eigenvalues of A. The magical thing is that they have the *same signs* as the eigenvalues, which is all you need to classify a quadratic form. Moreover it is *easy to find* D. You do *paired* row and column operations. Given a row operation, say $R_i \to R_i + kR_j$, there is a corresponding column operation $C_i \to C_i + kC_j$ in which you replace the i^{th} column of a matrix by its sum with k times the j^{th} column. In a similar fashion you have $C_i \leftrightarrow C_j$ and $C_i \to kC_i$, $k \neq 0$. Now, given a symmetric matrix start off as though you are trying to find the REF of A, BUT whenever you do a row operation IMMEDIATELY follow it by the corresponding column operation. So all your row and column operation will occur in pairs like this. The symmetry of the matrix tells you that you will eventually get a diagonal matrix at the end. This is your D. Different ways of organizing the computation may give you different diagonal entries, but you will always have the same number of positive entries, the same number of negative entries, and the same number of zero entries. This is called Sylvester's Law of Inertia.

See the class notes for examples of this. On the final, if you have a problem which requires merely the signs of the eigenvalues you may either find the eigenvalues the usual way or you may use the paired row and column method just described.