REVIEW FOR EXAM I

The exam covers sections 1.1–1.4, 2.1–2.4, and 3.1-3.3.

This review is basically a checklist; it does not cover every conceivable kind of problem that might be on the exam, but it does cover the main points. Neither is it the case that everything listed here will necessarily be on the exam. Look at the book and the class notes on D2L for details and examples.

Before getting into these matters here are some general remarks abut the nature of the exam. It has a lot of problems, but they are mostly rather short. You may have noticed that many of the problems in linear algebra reduce to finding and using the row echelon form of a matrix. In some of the problems on the exam you will be given the REF; it is then up to you to use it properly. So do not waste time finding an REF unnecessarily. On the other hand there will be some problems where you will need to find the REF or the RREF.

There will be problems where you are asked for some sort of numerical answer. Be sure that you know what *kind* of answer to give. Is it a number? Is it a vector? If it is a vector, then which $\mathbb{R}^n$ does it live in? Is it a matrix? If it is a matrix, then what is its shape? Is it a finite list of vectors? Is it an expression of a vector as a linear combination of vectors?

There will be problems where you are asked a question. Be sure to answer the precise question that was asked. If you are asked "Is this set linearly dependent?" and you think that it isn't, then say "No"; don't say something like "Yes, linearly independent." (You would be contradicting yourself.) Other problems will require True or False answers; be sure that you answer "True" or "False" rather than something else, like "Yes" or "No". Also, if a problem says that you do not have to give explanations, then you can just answer "Yes" or "No" or, respectively, "True" or "False".

Finally, remember that the exam is not WebAssign. In WebAssign you just entered final answers without showing how you got them. On the exam YOU WILL BE GRADED ON THE DETAILS OF THE PROCEDURE THAT YOU FOLLOWED TO GET THE ANSWER. This has two aspects to it. If you carry out a sequence of steps correctly but then make a mistake you may receive some credit for the correct steps you made. (But don't expect any credit beyond that point.) The second aspect is that if you write down a correct answer to a multistep problem without showing the steps you may receive no credit on the problem.

1. Vectors

   Given vectors $\mathbf{v}$ and $\mathbf{w}$ and a real number r, know how to compute the sum $\mathbf{v}+\mathbf{w}$ and scalar multiple $r\mathbf{v}$.

   Know how to compute the dot product $\mathbf{v}\cdot\mathbf{w}$; remember that this is a scalar NOT a vector! ($\mathbf{v}\cdot\mathbf{w} = v_1w_1 + v_2w_2 + \cdots + v_nw_n$ for $\mathbf{v},\mathbf{w}\in\mathbb{R}^{\mathbf{n}}$.)

Know how to compute the norm or magnitude $||\mathbf{v}|| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$.

Know that a unit vector $\mathbf{u}$ has $||\mathbf{u}|| = \mathbf{1}$ and that to find a unit vector $\mathbf{u}$ having the same direction as $\mathbf{v}$ you take $\mathbf{u} = \mathbf{v}/||\mathbf{v}||$.

Know that if θ is the angle between $\mathbf{v}$ and $\mathbf{w}$, then $\cos(\theta) = \mathbf{v} \cdot \mathbf{w}/(||\mathbf{v}||||\mathbf{w}||)$; know that $\mathbf{v}$ and $\mathbf{w}$ are orthogonal or perpendicular to each other if $\mathbf{v} \cdot \mathbf{w} = \mathbf{0}$.

Know that the projection $\mathbf{p}$ of $\mathbf{v}$ onto a vector $\mathbf{w}$ is given by $\mathbf{p} = \mathbf{proj_w v} = \left(\frac{\mathbf{w} \cdot \mathbf{v}}{\mathbf{w} \cdot \mathbf{w}}\right) \mathbf{w}$.

2. Force vectors

We covered force vectors in section 1.4.

Given a force vector $\mathbf{f}$ in $\mathbb{R}^2$ we can resolve it into a horizontal component $\mathbf{f}_x$ and a vertical component $\mathbf{f}_y$ such that $\mathbf{f} = \mathbf{f}_x + \mathbf{f}_y$, where $\mathbf{f}_x = [||\mathbf{f}|| \cos\theta, 0]$, $\mathbf{f}_y = [0, ||\mathbf{f}|| \sin\theta]$, and θ is the angle between the horizontal vector $[1, 0]$ and $\mathbf{f}$, measured counterclockwise from $[1, 0]$ to $\mathbf{f}$. (See the diagrams in the book.)

Resolutions are often used when we have several forces that act on an object, we know some of them, and we want to find the others, and we are given that the sum of the forces is zero, i.e. the object doesn't move. The idea here is to resolve all the forces into horizontal and vertical components. Then the sum of the horizontal components is zero, and the sum of the vertical components is zero. This may enable us to find the unknown components and thereby the unknown forces.

3. Lines and Planes

A line in $\mathbb{R}^2$ has normal equation $\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = \mathbf{0}$ and general equation $ax_1 + bx_2 = c$, where $\mathbf{p} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$ is a point on the line, $\mathbf{n} = \begin{bmatrix} a \\ b \end{bmatrix}$ is a vector perpendicular to the line, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, and $c = \mathbf{n} \cdot \mathbf{p}$. It has vector equation $\mathbf{x} = \mathbf{p} + t\mathbf{d}$ and parametric equations $x_1 = p_1 + td_1$ and $x_2 = p_2 + td_2$, where $\mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$ is a vector pointing in the same direction as the line.

A line in $\mathbb{R}^3$ has vector equation $\mathbf{x} = \mathbf{p} + t\mathbf{d}$ and parametric equations $x_1 = p_1 + td_1$, $x_2 = p_2 + td_2$, and $x_3 = p_3 + td_3$ where $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $\mathbf{p} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$ is a point on the line, and $\mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$ is a vector pointing in the same direction as the line.

Note that in both cases, if P and Q are points on the line, then one can take $\mathbf{p} = P$ and $\mathbf{d} = Q - P$.

A plane in $\mathbb{R}^3$ has normal equation $\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = \mathbf{0}$ and general equation $ax_1 + bx_2 + cx_3 = d$, where $\mathbf{p} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$ is a point on the plane, $\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is a vector perpendicular to the plane, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, and $d = \mathbf{n} \cdot \mathbf{p}$. It has vector equation $\mathbf{x} = \mathbf{p} + s\mathbf{u} + t\mathbf{v}$ and parametric equations $x_1 = p_1 + su_1 + tv_1$, $x_2 = p_2 + su_2 + tv_2$, and $x_3 = p_3 + su_3 + tv_3$, where $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ are non-zero vectors which are parallel to the plane but are not parallel to each other.

Note that if P, Q, and R are points on the plane which do not lie on the same line, then one can take $\mathbf{p} = P$, $\mathbf{u} = Q - P$ and $\mathbf{v} = R - P$.

4. Matrices

Know that an $m \times n$ matrix A has m rows and n columns; this is the shape of A. The i^{th} row is denoted by $\mathbf{A}_i$ (or by R_i if the matrix A is clear from the context). The j^{th} column is denoted by $\mathbf{a}_j$. The entry at the intersection of $\mathbf{A}_i$ and $\mathbf{a}_j$ is denoted by $a_{i,j}$ or sometimes by $[A]_{i,j}$.

Know how to compute the sum $A + B$ of two $m \times n$ matrices ($[A + B]_{i,j} = [A]_{i,j} + [B]_{i,j}$) and the scalar multiple rA of a matrix A by a number r ($[rA]_{i,j} = r[A]_{i,j}$).

The $m \times n$ zero matrix is the $m \times n$ matrix having all entries zero; it is denoted by O, where the shape is determined by the context. If A is an $m \times n$ matrix then $A + O = A = O + A$.

Know that the transpose A^T of an $m \times n$ matrix A is the $n \times m$ matrix A^T obtained by changing the rows to columns and the columns to rows, i.e. $[A^T]_{i,j} = [A]_{j,i}$); EVERY matrix has a transpose; it does NOT have to be a square matrix!

The product AB of an $m \times n$ matrix A and an $n \times p$ matrix B is the $m \times p$ matrix $AB = C$ ($[C]_{i,j}$ is the dot product of $\mathbf{A}_i$ and $\mathbf{b}_j$). Know that the product is not defined if the shapes of the matrices do not fit this pattern.

Know that for an $n \times n$ matrix A and a positive integer k the k^{th} power A^k is the result of taking the product of k copies of A.

Know what the $m \times m$ identity matrix I_m is ($[I_m]_{i,j}$ is 1 if $i = j$ and is 0 if $i \neq j$) and that if A is an $m \times n$ matrix, then $I_m A = A$ and $AI_n = A$; you should also know that an identity matrix is often written just as I, where the shape is determined by the context.

Know that $(AB)^T = B^T A^T$ and that, in general, $AB \neq BA$.

If A is a square matrix, i.e. it is $n \times n$, then $A^2 = AA$, $A^3 = AAA = A^2 A = AA^2$, etc. A^k is the k^{th} power of A. If A is not a zero matrix, the A^0 is defined to be I.

5. Row operations

Know how to do the three elementary row operations: $R_i \to R_i + kR_j$, $R_i \to kR_i$ (where $k \neq 0$), and $R_i \leftrightarrow R_j$. When you do these operations use this notation (not just a part of it!) to explain what you are doing. Label each row operation *before* you do it.

6. REF and RREF

A matrix in row echelon form (REF) looks like this:

$$\begin{bmatrix} 0 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 6 & 7 \\ 0 & 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Any zero rows are at the bottom. The first non-zero entry in a non-zero row is called the leading entry. As you move down the matrix the leading entries move to the right.

To put a matrix in REF you use Gauss reduction. Find the first non-zero column. Call this the current column. Mentally cross off any columns to the left of this column. If necessary, use row exchanges to make the top entry in this column non-zero; this is your first pivot. Then use row addition to make all the entries under this first pivot equal to zero. DO NOT TRY TO CREATE ZEROES ANYWHERE ELSE IN THE MATRIX UNTIL YOU FINISH THIS STEP. Now mentally cross off the row and column containing the first pivot. This will give you a smaller matrix (for example 3×4 instead of 4×5). Repeat the procedure with this smaller matrix, creating and using your second pivot. Continue until you get REF. Note that you will be working from left to right and top to bottom.

A matrix in reduced row echelon form (RREF) looks like this:

$$\begin{bmatrix} 0 & 1 & 0 & 4 & 0 \\ 0 & 0 & 1 & 6 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

It is in REF, all the leading entries are 1, and all the entries above a leading entry are 0.

There are two methods for putting a matrix in RREF.

In the first method you first put the matrix in REF using Gauss reduction. Then you use back reduction to put it in RREF: Mentally cross off any zero rows in the REF. Then, if necessary, use row multiplication to make your last leading entry equal to 1. Then use row addition to make all the entries above the last leading entry equal to zero. Mentally cross off the row containing the last leading entry to get a smaller matrix. Then repeat this process using the last leading entry of the smaller matrix. Continue until you get RREF. Note that this process works from right to left and bottom to top.

In the second method you use Gauss-Jordan reduction. It begins like Gauss reduction except for the fact that you make each pivot equal to 1 as you go and you make the entries both above and below each pivot equal to zero as you go. This process works from left to right. Gauss-Jordan at first seems easier, but in fact it requires around 50% more computational work than Gauss reduction plus back reduction.

Every non-zero matrix A has infinitely many different REF's. However it is a (non-obvious) fact that they all have the same number of non-zero rows. This number is called the rank of A, written $\text{rank}(A)$.

In contrast, a matrix has exactly one RREF. (Again this is not obvious.)

WARNING #1: Gauss reduction and Gauss-Jordan reduction are not just random sequences of row operations which somehow produce a matrix in REF or RREF. There are specific patterns to the sequences of row operations that you are supposed to follow. If you do not follow these patterns you may have points deducted even if you eventually get the right answer.

WARNING #2: If you make a numerical mistake I am not going to follow the chain of computations you do that results from that mistake. You will get no credit beyond the point at which you made the mistake.

7. Solving linear systems

To solve a system of m equations in n unknowns with coefficient matrix A and right hand side $\mathbf{b}$, i.e. $A\mathbf{x} = \mathbf{b}$, form the augmented matrix $[A|\mathbf{b}]$, then find its REF $[U|\mathbf{c}]$. This corresponds to the system $U\mathbf{x} = \mathbf{c}$. Note that U will automatically be an REF for A. The variables $x_1, x_2, \ldots, x_n$ correspond to the columns of U. Those which correspond to columns which contain leading entries are bound variables; the others are free variables. Note in particular that if U has a column of zeroes, then the corresponding variable is a free variable; DO NOT SET IT EQUAL TO ZERO; THERE IS NO REASON TO DO THIS!!!

The number of non-zero rows of U is the rank of A; call this number r. Note that the number of bound variables is r, and the number of free variables is $n - r$. The number of non-zero rows of $[U|\mathbf{c}]$ is the rank of $[A|\mathbf{b}]$; call this number r'. There are only two possibilities for the relationship between r' and r; either $r' = r$ (when $[U|\mathbf{c}]$ and U have the same number of non-zero rows) or $r' = r + 1$ (when $[U|\mathbf{c}]$ has exactly one more non-zero row than U).

There are now three possibilities for the number of solutions of $A\mathbf{x} = \mathbf{b}$:

Case 1: If $r' = r+1$, then there is no solution since the last non-zero row of $[U|\mathbf{c}]$ corresponds to the equation $0 = c_{r+1}$, where $c_{r+1} \neq 0$.

Case 2: If $r' = r$ and $r = n$, then there is a unique solution. The first condition implies that we don't have zero equalling something non-zero. The second condition says that there are no free variables; so every variable is a bound variable. The last non-zero row corresponds to an equation of the form $u_{r,n}x_n = c_r$; we solve this for x_n. The next to last non-zero row corresponds to an equation of the form $u_{r-1,n-1}x_{n-1} + u_{r-1,n}x_n = c_{r-1}$. We plug in the value of x_n that we just found and then solve for x_{n-1}. We then continue this process of back substitution until we have values for all of $x_1, x_2, \ldots, x_n$, giving exactly one solution.

Case 3: If $r' = r$ and $r < n$, then there are infinitely many solutions. Again the first condition says that we do have at least one solution. The second condition says that we have $n - r$ free variables. We set each of these equal to a parameter (s, t, u, etc.). Note that we must use a *different* parameter for each free variable. Then we solve the system by back substitution in a manner analogous to that in Case 2; you still solve the last equation (the one corresponding to the last non-zero row) first, then the next to last and so on. The difference now is that the free and bound variables are mixed together. For example, if x_n is bound, you solve for it and get a number. Then if x_{n-1} is free, you set it equal to a parameter, say t. Then if x_{n-2} is bound, you substitute into the next to last equation the number you found for x_n and the parameter t for x_{n-1} and then solve for x_{n-2}, and so on.

You may want to express the solution in vector form, e.g. if $x_1 = 3+s-t$, $x_2 = s$, and $x_3 = t$, then

$$\mathbf{x} = \begin{bmatrix} 3 & +s & -t \\ & s & \\ & & t \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} s \\ s \\ 0 \end{bmatrix} + \begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Note that if you know the numbers m, n, r, and r', then you can definitely determine how many solutions there are by looking at the three cases above.

Even if you don't know all these numbers you can sometimes get at least partial information on the number of solutions. For example, if you know that the right hand side of the original system is $\mathbf{b} = \mathbf{0}$, then you know that there will be at least one solution since $\mathbf{x} = \mathbf{0}$ is a solution, so the possibilities are unique solution or infinitely many solutions. As another example, if you know m and n, note that you always have $r \le m$, $r \le n$, $r' \le m$, and $r' \le n+1$. So if $n > m$, this tells you that $n > r$, so there are some free variables, hence it is impossible to have a unique solution; the possibilities are then either no solution or infinitely many solutions. In summary, think about *all* the possibilities for how r and r' are related and how r and n are related.

8. Problems about sets of vectors

Suppose we have a set of vectors $S = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k\}$ in $\mathbb{R}^m$. To do the following kinds of problems create a matrix A whose *columns* are $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k$. For example, if $S = \{[1,2,3],[4,5,6],[7,8,9],[10,11,12]\}$, then

$$A = \begin{bmatrix} 1 & 4 & 7 & 10 \\ 2 & 5 & 8 & 11 \\ 3 & 6 & 9 & 12 \end{bmatrix}.$$

Is $\mathbf{b}$ in the span of S? I.e. is $\mathbf{b} \in \text{span}(S)$? This is asking whether $\mathbf{b}$ is a linear combination $r_1\mathbf{w}_1 + r_2\mathbf{w}_2 + \cdots + r_k\mathbf{w}_k$ of the elements of S. This is equivalent to asking whether $A\mathbf{x} = \mathbf{b}$ has a solution. Answer this question using the methods in the previous section. Set up $[A|\mathbf{b}]$. It will have REF $[U|\mathbf{c}]$. If there is a non-zero row of $[U|\mathbf{c}]$ whose entries in U are all zero, then the answer is no. If there is no such row, then the answer is yes. In this case if the problem asks you to express $\mathbf{b}$ as a linear combination of the elements of S you should find *specific numbers* for $r_1, r_2, \ldots, r_k$ using your solution to $A\mathbf{x} = \mathbf{b}$ and then write down a specific linear combination, e.g. $\mathbf{b} = 2\mathbf{w}_1 - 7\mathbf{w}_2 + 4\mathbf{w}_3$.

Does S span $\mathbb{R}^m$? I.e. does $\text{span}(S) = \mathbb{R}^m$? This is asking whether $A\mathbf{x} = \mathbf{b}$ has a solution for *every* $\mathbf{b}$ in $\mathbb{R}^m$. If the REF U of A does not have a row of zeroes, the answer is yes, since we can always solve $U\mathbf{x} = \mathbf{c}$ and hence $A\mathbf{x} = \mathbf{b}$. If U does have a row of zeroes then the answer is no, since we can choose $c_{r+1} \neq 0$ and then work back to a vector $\mathbf{b}$ which is not in $\text{span}(S)$. Note that in this case you should be able to find a *specific* vector which is not in span(S) as follows. The last non-zero row of $[U|\mathbf{c}]$ will have the form $[0, 0, \ldots, 0, c_{r+1}]$, where c_{r+1} is some expression in the components $b_1, b_2, \ldots, b_m$ of $\mathbf{b}$. Just choose $\mathbf{b}$ so that $c_{r+1} \neq 0$. For example, if $m = 3$ and the last row of the REF is $[0, 0, 0, b_1 + 3b_2 - 2b_3]$, you can choose, for example, $b_1 = 1$, $b_2 = 0$, and $b_3 = 0$ to get $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

Is S linearly dependent? This is asking whether there is some linear combination of the elements of S which equals zero but for which at least one of the coefficients $r_i \neq 0$. This is equivalent to asking whether $A\mathbf{x} = \mathbf{0}$ has a non-trivial solution $\mathbf{x} \neq \mathbf{0}$. (Note again that this system always has the solution $\mathbf{x} = \mathbf{0}$; the question is whether there are any *other* solutions.) The augmented matrix is $[A|\mathbf{0}]$ has REF $[U|\mathbf{0}]$, so the whole question is whether there are any free variables. If there are some free variables, then the answer to the question is yes. In this case if you are asked to give a dependence relation you will need to find a *specific* non-trivial solution; you do this by setting at least one of your free variables equal to a non-zero number and solving for $\mathbf{x}$, then using the components of $\mathbf{x}$ as your r_i's. For example, if $[U|\mathbf{0}] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$, then you would set $x_3 = t$ and solve for $x_2 = -4t$ and $x_1 = 1 - 5t$; choose, for example, $t = 1$ to get $x_1 = -4$, $x_2 = -4$, and $x_3 = 1$. Then for your dependence relation write $-4\mathbf{w}_1 - 4\mathbf{w}_2 + \mathbf{w}_3 = \mathbf{0}$. On the other hand, if there are no free variables, then the answer is no, since then our system has only the trivial solution.

Is S linearly independent? This is asking whether S is *not* linearly dependent, and so is the same as asking whether the answer to the previous question is no.

9. Inverses

$n \times n$ matrices A and B are inverses of each other if $AB = I$ and $BA = I$. It is an easy fact that if C is a matrix with $AC = I$ and $CA = I$, then $C = B$, i.e. inverses are unique, so we may unambiguously write $B = A^{-1}$. In this case we say that A is invertible or non-singular. It is a (non-easy) fact that if one of $AB = I$ or $BA = I$ is true, then so is the other.

If A and B are invertible, then so is AB, and $(AB)^{-1} = B^{-1}A^{-1}$; note that the order of the matrices is reversed when taking inverses.

In general, to find the inverse of A you put the matrix $[A|I]$ in RREF. If you get $[I|B]$, then $B = A^{-1}$. Note that in this case the sequence of row operations $\rho_1, \rho_2, \ldots, \rho_k$ which takes $[A|I]$ to $[I|A^{-1}]$ will, when restricted to A, give a sequence of row operations

$$A \overset{\rho_1}{\to} A_1 \overset{\rho_2}{\to} A_2 \cdots A_{k-1} \overset{\rho_k}{\to} A_k = I$$

If the RREF does not have the form $[I|B]$, then A does not have an inverse, and we say that A is non-invertible or singular.

I call the procedure $[A|I] \to [I|A^{-1}]$ the "magic trick". Note that you should NOT leave your answer in the form $[I|A^{-1}]$; you should actually say what A^{-1} is.

The problem will be rigged so that it takes very few row operations to do.

There is a special case in which there is an easier way to compute the inverse:

A 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible if and only if $ad - bc \neq 0$. In this case

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

This is called the adjoint formula for the inverse. You will be expected to simplify your answer, i.e. calculate $ad-bc$ and carry out the scalar multiplication of the matrix by $1/(ad-bc)$ and simplify any resulting fractions.

10. Elementary matrices

Recall the three elementary row operations: $R_i \to R_i + kR_j$, $R_i \to kR_i$ (where $k \neq 0$), and $R_i \leftrightarrow R_j$. Remember when you do these operations use this notation to explain what you are doing. An elementary matrix E is the result of doing *one* elementary row operation to the identity matrix I. Warning: If you compute an elementary matrix E_1 and then need to compute a second elementary matrix E_2, do your second row operation on I, NOT on E_1. The basic fact about elementary matrices is that EA is the same matrix as the result of doing the corresponding elementary row operation on A.

Here are some 3×3 examples:

$R_3 \to R_3 + 4R_1$ gives $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}$.

$R_2 \to 2R_2$ gives $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

$R_2 \leftrightarrow R_3$ gives $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

To find the inverse of an elementary matrix E just do the row operation on I which "undoes" the row operation associated to E:

$R_i \to R_i - kR_j$ undoes $R_i \to R_i + kR_j$, e.g. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$.

$R_i \to (1/k)R_i$ undoes $R_i \to kR_i$, e.g. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

$R_i \leftrightarrow R_j$ undoes itself, e.g. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

The fact that $(AB)^{-1} = B^{-1}A^{-1}$ can be used to express A as a product of elementary matrices when A is invertible: Suppose we have a sequence of row operations $\rho_1, \rho_2, \ldots, \rho_k$ taking A to I.

$$A \xrightarrow{\rho_1} A_1 \xrightarrow{\rho_2} A_2 \cdots A_{k-1} \xrightarrow{\rho_k} A_k = I$$

Let the corresponding elementary matrices be $F_1, F_2, \ldots, F_k$. Then we have $A_1 = F_1A$, $A_2 = F_2A_1 = F_2F_1A$, ..., $A_k = F_kA_{k-1} = F_kF_{k-1}\cdots F_2F_1A$. Since $A_k = I$ we have $F_kF_{k-1}\cdots F_2F_1A = I$. If we then multiply on the left by $(F_kF_{k-1}\cdots F_2F_1)^{-1}$ (remember that matrix multiplication is not commutative, so we must choose a side on which to multiply) we get

$$(F_kF_{k-1}\cdots F_2F_1)^{-1}(F_kF_{k-1}\cdots F_2F_1)A = (F_kF_{k-1}\cdots F_2F_1)^{-1}I$$

$$IA = (F_kF_{k-1}\cdots F_2F_1)^{-1}$$

$$A = F_1^{-1}F_2^{-1}\cdots F_{k-1}^{-1}F_k^{-1}$$

So we have that $A = E_1E_2\cdots E_{k-1}E_k$ where $E_i = F_i^{-1}$.

Here is an example. Suppose we apply the sequence of row operations $\rho_1 : R_3 \to R_3 + 4R_1$, $\rho_2 : R_2 \to 2R_2$, $\rho_3 : R_2 \leftrightarrow R_3$ to the 3×3 matrix A to get the identity matrix I.

$F_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}$, $F_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, and $F_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

$$F_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix},\ F_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix},\ \text{and } F_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

$A = F_1^{-1}F_2^{-1}F_3^{-1}$ then gives the factorization

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

THIS IS THE WAY YOU SHOULD WRITE YOUR FINAL ANSWER: SPECIFIC MATRICES WITH NUMBERS IN THEM, WRITTEN SIDE BY SIDE IN THE CORRECT ORDER.

11. The big theorem on inverses

Let A be an $n \times n$ matrix. (The matrix MUST be square for this to work.) Then the following statements are equivalent.

(a) A is invertible.

(b) $A\mathbf{x} = \mathbf{b}$ always has a unique solution.

(c) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(d) The RREF of A is I.

(e) A is a product of elementary matrices.

Note: There are several more statements one could add to this list. Think through the possibilities.

12. Applications

We have studied one main application of the material so far: electric circuits consisting of resistors and batteries. This gives a graph (dots connected by line segments); the dots are called vertices and the line segments are called edges. The dots are where wires are joined together. The line segments then go between dots (including perhaps back to the same dot.) One gives a sense of direction to each edge. This allows us to measure the current I (in amps) flowing along that edge. It may be positive, negative, or zero. Using a volt meter we can measure the voltage drop across each component in that edge; put the positive lead on the "upstream" side of the component and the negative lead on the downstream side. For a resistor with resistance R the voltage drop is IR. For a battery of voltage V the voltage drop is V if the current is flowing through the battery from the positive terminal to the negative terminal. It is $-V$ if it is flowing in the opposite direction through the battery. The sum of the voltage drops around each loop is equal to zero. In addition the sum of the currents flowing into a node equals the sum of the currents flowing out of the node. This gives a system of linear equations which can then be solved to find the currents.