1st Order Linear D.E.

General Comments:
Given the differential equation,
\[ y' + p(t)y = g(t) \]
the solution takes the form:
\[ y(t) = c_1 y_1 \]
where \( y_1 \) is obtained from using the integrating factor method.

Derivation:
Define the integrating factor as:
\[ \mu = e^{\int p(t)dt} \]
Observe that taking the derivative with respect to \( t \) yields:
\[ \mu' = \frac{d}{dt}(\int p(t)dt)e^{\int p(t)dt} = p(t)e^{\int p(t)dt} = p(t)\mu \]
Distributing the integrating factor through the differential equation reveals an internal
product rule on the left hand side of the equation.
\[ \mu(y' + p(t)y) = \mu g(t) \]
\[ \mu y' + (\mu p(t))y = \mu y' + \mu' y = \mu g(t) \]
\[ (\mu y)' = \mu g(t) \]
Integrating the sides with respect to \( t \) and dividing by the integrating factor solves the
differential equation
\[ \int (\mu y)'dt = \int \mu g(t)dt \]
\[ y\mu = \int \mu g(t)dt \]
\[ y = \frac{1}{\mu} \int \mu g(t)dt \]

Solution Steps:
\[ y' + p(t)y = g(t) \]
\[ \mu = e^{\int p(t)dt} \]
\[ y = \frac{1}{\mu} \int \mu g(t)dt \]
Exact Equations

**General Comments:**
Given the differential equation,

\[ M(x, y) + N(x, y) \frac{dy}{dx} = 0 \]

the method of exact equations can be used to solve for \( y(x) \)

**Testing Exactness:**
If the function is exact in its current state, then

\[ \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \text{ and } \frac{d\mu}{dx} = 0 \]

otherwise an integrating factor \( \mu \neq 0 \) is required. This integrating factor can take two forms depending on assumption that it is purely a function of \( x \) or \( y \):

\[ \frac{d\mu}{dx} = M_y - N_x \mu = -\frac{M_y - N_x}{M} \mu \]

which yields

\[ \frac{\partial \mu M}{\partial y} = \frac{\partial \mu N}{\partial x} \]

**Solution Method:**

\[ \mu(M(x, y) + N(x, y) \frac{dy}{dx}) = \mu M(x, y)dx + \mu N(x, y)dy = 0 \]

\[ \Psi(x, y) = \int (\mu M(x, y)dx + \mu N(x, y)dy) = \int 0dx = c \]

\[ \Psi(x, y) = \int \mu M(x, y)dx + \int \mu N(x, y)dy = c \]

**Removing Redundant Terms:**
The resulting solution equation \( \Psi \) must be such that, omitting constants \( C \),

\[ \Psi(x, y) = \int \mu M(x, y)dx = \int \mu N(x, y)dy \]

which allows for the reduction of redundant terms.
Tank Mixture Problems

General Comments:
Given a tank problem drawn below where \( x_i(t) \) is the amount of substance mixed with water, \( V_i \) is the volume of the tank, the solution is as follows:

Flow Diagram.png

General Differential Equation:
For the tank system, a differential equation for each tank is written as the rate of mixture in minus the rate of mixture out. Note the concentrations (\( \gamma \)) and flow rate (\( \dot{V} \)) of each stream leaving and entering the tank.

\[
\frac{d(x_i)}{dt} = \Sigma(\gamma_i \dot{V_i})_{in} - \Sigma(\gamma_i \dot{V_i})_{out}
\]

Note that fluid leaving a "well stirred tank" has concentration \( \gamma_{well-stirred} = \frac{x_i(t)}{V_i} \)

Solution Method: Summing all the concentrations and flow rates in and out of tank 1 and two yields the following system of equations:

\[
\frac{d(x_1)}{dt} = \gamma_{in} \dot{V}_{in} + \gamma_{21} \dot{V}_{21} - \gamma_{out,1} \dot{V}_{out,1} - \gamma_{12} \dot{V}_{12}
\]

\[
\frac{d(x_2)}{dt} = \gamma_{12} \dot{V}_{12} - \gamma_{21} \dot{V}_{21} - \gamma_{out,2} \dot{V}_{out,2}
\]

Replacing the concentration terms with their well-stirred terms:

\[
\frac{d(x_1)}{dt} = \gamma_{in} \dot{V}_{in} + \frac{x_2}{V_2} \dot{V}_{21} - \frac{x_1}{V_1} \dot{V}_{out,1} - \frac{x_1}{V_1} \dot{V}_{12} = \gamma_{in} \dot{V}_{in} + \frac{x_2}{V_2} \dot{V}_{21} - \frac{x_1}{V_1} \left( \frac{\dot{V}_{out,1} + \dot{V}_{12}}{V_1} \right)
\]

\[
\frac{d(x_2)}{dt} = \frac{x_1}{V_1} \dot{V}_{12} - \frac{x_2}{V_2} \dot{V}_{21} - \frac{x_2}{V_2} \dot{V}_{out,2} = \frac{x_1}{V_1} \dot{V}_{12} - \frac{x_2}{V_2} \left( \frac{\dot{V}_{out,2} + \dot{V}_{21}}{V_2} \right)
\]

Note: If the tank system has only one tank, only the first equation is required with \( \dot{V}_{21} = \dot{V}_{12} = 0 \)
Tank Mixture Problems (Cont.)

Elimination Method:
Similar to how linear systems of equations were solved in College Algebra, adding and subtracting the differential equations here will be used to reduce the equation to a function of only one variable.

First each equation is set equal to 0:

\[
\frac{d(x_1)}{dt} = \frac{\dot{V}_{in}}{V_1} + x_2 \frac{\dot{V}_{21}}{V_2} - x_1 \left( \frac{\dot{V}_{out,1} + \dot{V}_{12}}{V_1} \right) \rightarrow \frac{d(x_1)}{dt} - \frac{\dot{V}_{in}}{V_1} = 0
\]

\[
\frac{d(x_2)}{dt} = \frac{x_1 \dot{V}_{12}}{V_1} - x_2 \left( \frac{\dot{V}_{out,2} + \dot{V}_{21}}{V_2} \right) \rightarrow \frac{d(x_2)}{dt} - \frac{x_1 \dot{V}_{12}}{V_1} = 0
\]

Let \( D = \frac{d}{dt} \) such that \( D[x_i] = \frac{dx_i}{dt} \) and factor out the \( x_1, x_2 \) terms as follows:

\[
(D + \left( \frac{\dot{V}_{out,1} + \dot{V}_{12}}{V_1} \right))[x_1] - \dot{V}_{in} = 0
\]

\[
(D + \left( \frac{\dot{V}_{out,2} + \dot{V}_{21}}{V_2} \right))[x_2] - \frac{x_1 \dot{V}_{12}}{V_1} = 0
\]

Multiplying equation one by \( \frac{\dot{V}_{12}}{V_1} \) and equation two by \( (D + \left( \frac{\dot{V}_{out,1} + \dot{V}_{12}}{V_1} \right)) \) allows us to cancel the \( x_1 \) terms through addition of the two equation and results in the differential equation:

\[
\left(D + \frac{\dot{V}_{out,1} + \dot{V}_{12}}{V_1}\right)\left(D + \frac{\dot{V}_{out,2} + \dot{V}_{21}}{V_2}\right)\left[x_2\right] - \frac{\dot{V}_{in}}{V_1} \left(\dot{V}_{in} + x_2 \frac{\dot{V}_{21}}{V_2}\right) = 0
\]

\[
\frac{d^2(x_2)}{dt^2} + \left( \frac{\dot{V}_{out,1} + \dot{V}_{12}}{V_1} + \frac{\dot{V}_{out,2} + \dot{V}_{21}}{V_2} \right) \frac{dx_2}{dt} + \frac{\left(\dot{V}_{out,1} + \dot{V}_{12}\right)(\dot{V}_{out,2} + \dot{V}_{21}) - \dot{V}_{12} \dot{V}_{21}}{V_1 V_2} \frac{x_2}{\dot{V}_{in} V_1} = \frac{\dot{V}_{12}}{V_2} \frac{\dot{V}_{in} V_1}{V_1}
\]

\( x_2 \) can be solved by use of the following methods of solving second order non-homogeneous differential equations.

Further, \( x_1 \) can be solved for by another elimination method or by substituting \( x_2, x_2' \) back into the above differential equation.
Homogeneous Equations

General Comments:
Given the differential equation,
\[ \frac{d^2y}{dt^2} + p(t) \frac{dy}{dt} + q(t)y = 0 \]
the solution takes the form:
\[ y(t) = c_1y_1 + c_2y_2 \]
where \( y_1 \) and \( y_2 \) are obtained by solving the characteristic equation of the differential equation.

Characteristic Equation:
If \( p(t) \) and \( q(t) \) are constants, assume a solution of \( y = e^{λt} \) with first and second derivatives \( λe^{λt} \) and \( λ^2e^{λt} \) respectively.

Plugging this into the differential equation yields:
\[ e^{λt}(λ^2 + p(t)λ + q(t)) = 0 \]
which is solved through the quadratic formula to yield homogeneous solutions \( y_1 \) and \( y_2 \).
\[ λ_{1,2} = \frac{-p(t) ± \sqrt{p(t)^2 - 4q(t)}}{2} \rightarrow y_1 = e^{λ_1t}, \ y_2 = e^{λ_2t} \]

Repeated Roots:
If \( λ_1 = λ_2 \) then the repeated homogeneous solution is multiplied by \( t \) such that
\[ y_1 = e^{λt}, \ y_2 = te^{λt} \]

Complex Roots:
If \( p(t)^2 - 4q(t) < 0 \) such that the roots of the characteristic equation are complex, they take the form:
\[ λ_{1,2} = \frac{-p(t)}{2} ± \frac{\sqrt{4q(t) - p(t)^2}}{2}i = η ± ωi \rightarrow y_1 = e^{(η+ωi)t}, \ y_2 = e^{(η-ωi)t} \]
by the Euler identity \( e^{ia} = \cos(α) + i\sin(α) \) and rearranging of the variables, the solution becomes
\[ y_1 = e^{ηt}\cos(ωt), \ y_2 = e^{ηt}\sin(ωt) \]
General Comments:
Given the differential equation,
\[
\frac{d^2y}{dt^2} + p(t) \frac{dy}{dt} + q(t)y = g(t)
\]
the solution takes the form:
\[
y(t) = c_1y_1 + c_2y_2 + y_p
\]
where \(y_1\) and \(y_2\) are obtained by solving the differential equation as a homogeneous equation and method of undetermined coefficients is utilized to obtain \(y_p\).

For the method of undetermined coefficients, \(y_p\) is defined based off of \(g(t)\) and the solutions to the homogeneous case.

Choosing \(y_p\)’s Form from \(g(t)\):
- If \(g(t)\) has a term \(t^n\), then \(y_p\) should be assumed to contain a polynomial of degree \(n\) such that \(y_p = (A_n t^n + A_{n-1} t^{n-1} + \ldots + A_0 t^0) \ast h(t)\) where \(h(t)\) is any other functions from this section.

- If \(g(t)\) has a term \(\sin(\omega t)\) or \(\cos(\omega t)\), then \(y_p\) should be assumed as \(y_p = (A_0 \cos(\omega t) + A_1 \sin(\omega t)) \ast h(t)\) where \(h(t)\) is any other functions from this section.

- If \(g(t)\) has a term \(e^{\alpha t}\), then \(y_p\) should be assumed as \(y_p = A_0 e^{\alpha t} \ast h(t)\) where \(h(t)\) is any other functions from this section.

Choosing \(y_p\)’s Form from Homogeneous Solutions:
- If the proposed solution created from the above \(g(t)\) rules has a term that matches the form of any solutions to the homogeneous equation, the anticipated solution should be multiplied by \(t\).

Solution of Coefficients:
Take all required derivatives of the proposed solution and plug in to the differential equation:
\[
y_p'' + p(t)y_p' + q(t)y_p = g(t)
\]
and solve for all constants in \(y_p\).
Nonhomogeneous Equations - Variation of Parameters

**General Comments:**
Given the differential equation,

\[
\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = g(t)
\]

the solution takes the form:

\[
y(t) = c_1y_1 + c_2y_2 + y_p
\]

where \(y_1\) and \(y_2\) are obtained by solving the differential equation as a homogeneous equation and variation of parameters is utilized to obtain \(y_p\).

**Particular Solution:**
For variation of parameters, \(y_p\) is defined:

\[
y_p = \nu_1y_1 + \nu_2y_2, \quad \nu_1 = \int -\frac{g(t)y_2}{W} dt, \quad \text{and} \quad \nu_2 = \int \frac{g(t)y_1}{W} dt
\]

*Note that \(\nu_1\) includes a negative sign but \(\nu_2\) does not.*

with \(W\) in this case being the Wronskian.

The Wronskian can be calculated by the following formula:

\[
W = det\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = y_1y_2' - y_2y_1'
\]

*Note that the Wronskian cannot be zero as used in the integration above.*

**Reduction of Order:**
If \(y_1(t)\) is not identically 0, then the homogeneous solutions to the differential equation are related by

\[
y_2 = y_1 \int e^{-\int p(t)dt} \frac{y_2'}{y_1^2} dt
\]
Laplace Transforms

General Comments:
Given the differential equation and initial conditions,
\[
\frac{d^2 y}{dt^2} + p(t) \frac{dy}{dt} + q(t)y = g(t), \quad y(0) = k_1, \quad y'(0) = k_2
\]
the solution takes the form:
\[
y(t) = c_1 y_1 + c_2 y_2 + y_p
\]
where \(y_1\), \(y_2\), and \(y_p\) are obtained by transforming the differential into the Laplace domain and returning the solution to the time domain.

Properties of the Laplace Transform:
- The Laplace Transform is a linear operator which means it can be "distributed" over terms being added or subtracted. I.e.
\[
\mathcal{L}[A(t) + B(t)] = \mathcal{L}[A(t)] + \mathcal{L}[B(t)]
\]
- Scalar multiples of functions undergoing Laplace Transforms are as follows:
\[
\mathcal{L}[cA(t)] = c\mathcal{L}[A(t)]
\]

Laplace Transform of \(y^{(n)}(t)\):
The Laplace Transform of any derivative of the solution function yields,
\[
\mathcal{L}[y^{(n)}(t)] = s^n Y(s) - s^{n-1} y(0) - s^{n-2} y'(0) - \ldots - y^{n-1}(0)
\]
which, for second order differential equations, means
\[
\mathcal{L}[y(t)] = Y(s), \quad \mathcal{L}[y'(t)] = sY(s) - y(0), \quad \mathcal{L}[y''(t)] = s^2 Y(s) - sy(0) - y'(0)
\]

Second Order D.E. Solution:
(Assuming \(p(t)\) and \(q(t)\) are constants where \(p(t)=p\) and \(q(t)=q\))
\[
Y(s) = \frac{\mathcal{L}[g(t)] + (s + p)y(0) + y'(0)}{s^2 + ps + q}
\]
\[
y(t) = \mathcal{L}^{-1}[Y(s)] = \mathcal{L}^{-1}\left[\frac{\mathcal{L}[g(t)] + (s + p)y(0) + y'(0)}{s^2 + ps + q}\right]
\]

Laplace Transform Table:
A table of common Laplace Transforms and their inverses is located in the textbook.
Laplace Transforms (Cont.)

Unit Step Functions and Impulse Functions:
Unit step functions and impulse functions are piece-wise inputs to a system modelled in a differential equation. They are defined:

\[
u(t-c) = u_c = \begin{cases} 
0 & t < c \\
1 & c \leq t 
\end{cases}
\]

\[
\delta(t-c) = \delta_c = \begin{cases} 
0 & t \neq c \\
1 & t = c 
\end{cases}
\]

Properties of Laplace Transform Revisited:
Given a Laplace Transform of a function multiplied by a unit step function or impulse function, the transform has the property:

\[
\mathcal{L}[u_c f(t-c)] = e^{-cs} \mathcal{L}[f(t)] = e^{-cs}[F(s)]
\]

\[
\mathcal{L}[k \delta_c] = ke^{-cs}
\]

Note that an impulse does not have a function multiplied by it, but can have a magnitude

Inverse Laplace Transform of Unit and Impulse Functions:

\[
\mathcal{L}^{-1}[e^{-cs} f(s)] = u_c \mathcal{L}^{-1}[F(s)] = u_c f(t-c)
\]